Dramatic Connections

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Stories are remembered without review, say the cognitive psychologists. Stories have self-referencing cues. Whereas none of the following are stories, there are "scenes" involving social situations.

When these are presented in class, students often shake their heads in disbelief. This writer (their teacher) is quick to confess that many mathematics educators are incensed by such approaches to teaching and learning -- "You're not supposed to teach mathematics like THAT!"

Even such individuals, however, will stoop to using the first example as a memory aid for trigonometry students.

<u>SOH CAH TOA</u> "Sore foot? Soak a toe-a!"

$$\text{sine } \theta = \frac{\text{Opposite side}}{\textit{Hypotenuse}} \qquad \text{cosine } \theta = \frac{\text{Adjacent side}}{\textit{Hypotenuse}} \qquad \text{tangent } \theta = \frac{\text{Opposite side}}{\text{Adjacent side}}$$

<u>Doomed Digits in Rounding</u>

A. ***Whole Numbers*** Suppose we need to round 6,782 to the nearest hundred. The student is asked, "Which one is the hundreds digit?" After correct response, the student is told that the seven is the <u>*key*</u> digit, and is shown to mark it with an arrow, as indicated below:

$$6,\overset{\downarrow}{7}82$$

The teacher then announces that all digits to the right of the key digit are **doomed**, that the doomed digits will turn into zeroes, and so the choice of the word "doomed" has been proper.

The eight, or the ten's digit, is then designated as the "**first doomed digit**" and is used to decide the following:

"Do we add zero $\begin{array}{l} 6,\overset{\downarrow}{7}82 \\ \underline{+\ 0} \end{array}$ or do we add one $\begin{array}{l} 6,\overset{\downarrow}{7}82 \\ \underline{+\ 1} \end{array}$ to the key digit?"

B. ***Decimal Fractions*** (**super-doomed** digits) This can be explained much as rounding whole numbers, except that **doomed** digits are **"super-doomed"** -- they <u>disappear entirely</u> -- for reasons related to accuracy and significant digits.

Example: if .345 is rounded to the nearest tenth, the correct answer is .3 . When .300 is written, the convention is that an actual measurement was made out to the nearest thousandth, and no hundredths were found, nor were thousandths found. Writing .3 when rounding .345 to the nearest tenth does not erroneously imply more accuracy than what really exists.

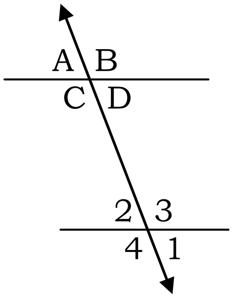
The % symbol and the decimal point are good friends

A. ***Changing per cents to decimal fractions*** "The decimal point and percent sign are **good friends**; they have conspired against math students and teachers for years to cause trouble. Because they are such friends, when the percent sign is chopped off, the sorrowful decimal point moves two places away from where his friend was."

Such is intended to help the student recall. Such mnemonics are often scorned by those who believe that understanding equates to recall.
While understanding is important, it does not contribute to recall.

B. ***Changing decimal fractions to per cents*** "But when his friend returns, the happy decimal point moves two places toward his long-lost friend."

Alternate interior angles: inside sandwich, on alternate sides of toothpick.



In the illustration, the horizontal segments represent fragments of bread held together by an intersecting line that works as a toothpick, as for an hors d'oeuvre.

$\angle C$ and $\angle 3$ are inside the sandwich and on opposite sides of the toothpick. Consequently, $\angle C$ and $\angle 3$ are alternate interior angles. Likewise, $\angle 2$ and $\angle D$ are also alternate interior angles.

$\angle A$ and $\angle 1$ are outside the sandwich and on opposite sides of the toothpick. Consequently, $\angle A$ and $\angle 1$ are alternate exterior angles. Likewise, $\angle B$ and $\angle 4$ are also alternate exterior angles.

GOO and LAB -- many absolute value expressions in inequalities

A. For $|2x+6|<4$ --

- With the variable term on the left, the statement is read "**L**ess than."
- The solution is $-5<x<-1$. The solution statement strictly includes "**A**nd" : $-5<x$ AND *at the same time*, $x<-1$. The "and" is normally omitted in the solution statement.
- For a LESS-than inequality involving AND in the solution statement, the graphed solution is a segment, resembling a **B**ARBELL.

B. For $|2x+11|\geq 5$--

- With variable term on the left, the statement is read "**G**reater than."
- The solution is $x\geq -3$ Or $x\leq -8$.
- For a **G**REATER-than inequality involving **O**R in the solution statement, the graphed solution resembles **O**ars of infinite length. The grips of the oars and the endpoints of the two opposite rays.

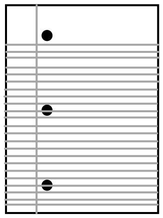
	Less than	**G**reater-than
Patterns?	**A**nd	**O**r
	Barbell	**O**ars of infinite length

Anti-Establishment Hippie Rebellious Contrarian Solving Strategy -- regarding steps in solving linear equations, and planting a global notion.

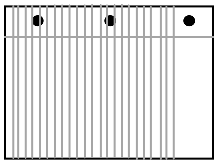
Equation	What is **seen** / “guesswork”	What is **done** to solve	OR
$x-3=8$	subtraction	addition	
$x+7=-15$	addition	subtraction	adding the opposite
$6x=30$	multiplication	division	multiplication by reciprocal
$\frac{x}{5}=9$	division	multiplication	**INVERSE OPERATIONS**

Notice the contrast between the second and third columns. There is an “anti” tone here, as if you’re told to **go** at a red light, or unwrapping a package. This is helpful when two-stage solving involves order of operations, which will also suggest rebellious, contrary orneriness. The 1960’s offered this ornery statement:

If they give you a piece of paper with lines on it...



..turn it sideways...



...and write the other way.

If you understand that thought, then you should be very good at algebra.

“Guesswork” ? In $2x+3=17$, an unknown number is being multiplied by two and then three is added to that product. If someone were to try to solve this equation by guessing ($x=10$? or $x=-5$?), the order of operations would be followed. **TRY 10 --**

Try 10: $2\times 10+3$

first, multiply: $20+3$

last, add: 23 → NO GOOD! not 17!

Try –5:

Try -5: $2\times(-5)+3$

first, multiply : $-10+3$

last, add : -7 → STILL NO GOOD!

Solving steps are **ultimate rebellion**: *inverse operations in reverse order*. Instead of first-multiply-last-add, solve by first-subtract-last-divide.

Solve $2x+3=17$

$$\left\{\begin{array}{ll} 2x+3=17 & \text{instead of adding last,} \\ \underline{\;\;-3\;\;\;-3} & \text{anti-add first} \\ 2x=14 & \text{instead of multiplying first,} \\ \frac{2x}{2}=\frac{14}{2} & \text{anti-multiply last} \\ x=7 & \end{array}\right.$$

Sine is friendly and a good mixer; Cosine is stuck-up and unfriendly.

The four trigonometric identities at the right are the basis for development of several other identities. The four shown here must be learned. It looks tough. As with many related ideas, it may be easier to learn the collection than to learn the individual parts. Several students have agreed that the following social nonsense is very helpful here.

$$\sin(u+v) = \sin u \bullet \cos v + \sin v \bullet \cos u$$
$$\sin(u-v) = \sin u \bullet \cos v - \sin v \bullet \cos u$$
$$\cos(u+v) = \cos u \bullet \cos v - \sin u \bullet \sin v$$
$$\cos(u-v) = \cos u \bullet \cos v + \sin u \bullet \sin v$$

- The first two identities are about sine. The results on the right have two terms, each including a sine and a cosine factor -- a **friendly**, social arrangement. Further, when the initial sine expression on the left involves addition, the right side also has addition. Subtraction is likewise **consistent**.
- The last two identities are about cosine. The results on the right have two terms. The first term has only cosine factors -- an **unfriendly**, **stuck-up**, snotty arrangement. The sine factors are collected at the end by themselves. Further, when the initial cosine expression on the left involves addition, the right side has subtraction. When the initial cosine expression on the left involves subtraction, the right side has addition -- likewise **inconsistent**.

Calculus: emancipation of u^2.

This nonsense also involves learning a collection instead of learning three individual parts. The order here of sine, tangent, and secant must be recalled.

- All three results have the same numerator.
- In the first denominator, u^2 is subjugated with respect to the 1, much like a dumped-on freshman.
- In the second denominator, u^2 is on par with the 1. The status of u^2 has improved. It is no longer under the radical as well -- sophomore?
- In the third denominator, the status of u^2 has surpassed that of the 1, relative to the first denominator. The bonus $|u|$ is not under the radical either -- u has come a long way. A junior? A senior?

$\frac{d}{dx}[\arcsin u] =$	$\frac{u'}{\sqrt{1-u^2}}$
$\frac{d}{dx}[\arctan u] =$	$\frac{u'}{1+u^2}$
$\frac{d}{dx}[\operatorname{arc\,sec} u] =$	$\frac{u'}{\lvert u\rvert\sqrt{u^2-1}}$

Clarifying the Varied Roles of Variables

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Most algebra teachers are secure with varied roles of variables, and have forgotten initial personal confusion. This initial student confusion is usually evident when the following occurs, if not before.

Having solved equations using one or two steps, students are ready to use commutative and associative properties to simplify equation members of three or more terms before solving. The teacher wants to inductively (developmentally) present the commutative property. The teacher shows several examples from arithmetic, writing and speaking.

"Three plus four is the same as four plus what?" With class help, the teacher writes "3 + 4 = 4 + ____" and then "3 + 4 = 4 + 3." Similar examples follow. Then the teacher climactically writes

"$x + y =$ ______" and with class help writes $\boldsymbol{x + y = y + x}$.

The teacher then shows how this property is used to simplify members of equations, so solving emphasis resumes. Soon, however, a student hand goes up. The student says, "Go back to the equation with both x's and y's. Should we solve for the x or the y?"

At this point, if not before, the teacher should begin to explain to the class that variables are used in many different ways in algebra. An organic classroom poster may be started, added to as the "roles of variables" list expands throughout the course. At the point of the discussion above, the list would include a few at the top of this list.

A. An unknown number in an equation used for solving practice: $x + 2 = 5 - x$
B. An idea in a word problem: x = the first odd integer
C. A specific idea in a formula: $A = \frac{bh}{2}$
D. A non-specific number in a property statement: $(x+y)+z = x+(y+z)$
E. A variable in an expression for simplification practice: "Simplify $x + x(x+1) + 3x + x^2$ = ?"
F. A constant: $ax^2 + bx + c = 0$
G. Substitution practice: "Evaluate $x^2y - \frac{x}{y}$ if $x = 4$ and $y = -2$.
H. The name of a set: $I = \{...-3, -2, -1, 0, 1, 2, 3,...\}$
I. Labels for axes, as the x-axis, y-axis, z-axis.

This poster would hang on the classroom wall and be added to as needed. For example, at some point later in the year, the teacher can makes further use of this issue as follows:

"Today we study an algebraic function. We label algebraic functions using alphabet letters to keep them separate. This may be confusing, so let's review our "Roles of Variables" poster and then add

J. Distinguishing one algebra function $y = f(x) = 2x + 7$ from another, $y = g(x) = 6 - x$ for ease in discussion and thought.

W.A.L.K.: We Add Like Kinds, Always

1. Whole number addition

Procedure: We line up whole numbers right-justified to add, keeping columns neat.

W.A.L.K.: We add ones together, tens together, hundreds together, and so on. When adding ones together produces new tens, the new tens are added to the given tens.

2. Arithmetic word problems

Procedure: We add pigs to pigs and cows to cows.

W.A.L.K.: We can add pigs to cows when they have a common name (animals).

3. Common fraction addition

Procedure: We add fractions having like denominators.

W.A.L.K.: Fractions with unlike denominators are given common denominators before addition is attempted.

4. Multiplication genesis

Procedure: We add groups of equal size: $7 + 7 + 7 = 3 \bullet 7$

W.A.L.K.: $7 + 7 + 7 + 5 + 5 = 3 \bullet 7 + 2 \bullet 5$

5. Decimal fraction addition

Procedure: We write numbers aligned at the decimal point, keeping columns neat.

W.A.L.K.: We add whole numbers together, tenths to tenths, hundredths together, and so on. When adding tenths produces new ones, the new ones are added to given ones, etc.

6. Radicals

Procedure: We add $\sqrt{2}+\sqrt{2}+\sqrt{2}=3\sqrt{2}$

W.A.L.K.: $2\sqrt{3}+4\sqrt{5}+3\sqrt{5}+8\sqrt{3}=10\sqrt{3}+7\sqrt{5}$

7. Similar terms

Procedure: We add $x^2+x^2+x^2=3x^2$

W.A.L.K.:
$$\left\{\begin{aligned} & x^2+x^2+x^2+x+x+x+4x+x^3+x^3 \\ & \qquad = 3x^2+7x+2x^3 \end{aligned}\right.$$

Arithmetic Catalysts for Algebra Word Problems

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Algebra word problems are rarely approached from the standpoint of arithmetic. For example, students are thrown into these types of word problems with almost no preview of the underlying arithmetic situations:

Consecutive integers
Distance-rate-time
Angles
Rate-of-work
Coin
Digit
Chemical mixture
Per cent mixture

An arithmetic approach may be helpful. An arithmetic background might begin with discussions and problems such as the following in the assignments several days prior to the standard problems:

For <u>*consecutive integers*</u>:

A. Integers are elements of the set {..., –3, –2, –1, 0, 1, 2, 3, ...} Three consecutive integers are 19, 20, 21. Four consecutive *odd* integers are 33, 35, 37, 39.

- Write the five consecutive *even* integers that precede 43.
- Represent four consecutive integers if the smallest is a.
- Represent three consecutive odd integers if the smallest is m.
- Represent five consecutive even integers if the smallest is p.

For <u>*angle problems*</u>:

B. Two complementary angles have sum of 90°; two supplementary angles have sum 180°. Find the measure of an angle that is complementary to a 34° angle; then find the supplement of the same angle. Similar exploration for angle sums in triangles should precede related word problems.

- An angle has measure of x°. Represent its complement.
- An angle has measure of y°. Represent its supplement.
- Two angles of a triangle have measure of z°. Represent the measure of the third angle.

For <u>*distance-rate-time*</u>:

C. How far does a plane travel in three and one-half hours at a rate of 460 miles per hour? How long does it take a cyclist to go 91 miles at 13 miles per hour? What is the rate of a swimmer who does $3a$ laps in $5x^2$ minutes?

For <u>*rate-of-work*</u>:

D. How much of a job gets done if someone does 1/8 of the job each hour and works for 5 hours?

E. What fraction of a job gets done if a person can finish the job in 20 minutes but only works for 12 minutes?

F. What fraction of a job gets done if two people work on it together for 20 minutes, one at a rate of 1/60 of the job per minute and the other at a rate of 1/45th of the job per minute?

G. What is the rate of work, per minute, for a machine that can finish a job in 13 hours?

H. What fraction of a job gets done if three people work on it for 15 minutes, the first able to finish alone in 60 minutes, the second in 80 minutes, and the third in 100 minutes?

H'. What fraction of a job gets done if three people work on it for b minutes, the first able to finish alone in x minutes, the second in y minutes, and the third in z minutes?

For *digit*:

I. If a number has 5 for a ten's digit and 8 for a one's digit, what is the number represented by the digits?

J. What arithmetic do we do to these digits to get the number?

K. What is the number with the digits reversed?

L. What is the difference between the original number and the reversed number?

M. What number is made if the ten's digit is x and the one's digit is y? Then what if these digits are reversed?

For *per cent mixture*:

N. If 20 pounds of beach sand contains 2 pounds of tar, what per cent of the 20-pound mix is tar?

O. If five pounds of pure sand is added, what per cent is tar?

P. If five pounds of pure tar is added to the original mix, what per cent is now tar?

A discussion of relevance is appropriate here, as many in mathematics education have suggested that these problem types are contrived and useless to students. Others have answered that each type described above seems to lend itself to learning some kinds of skills.

The latter position will be supported here. For example:

- The parallel-opposing forces situation in distance-rate-time problems gives students practice in setting up systems of equations to solve word problems, as well as exemplifying physics.
- Coin problems help students learn to distinguish relationships between quantities of represented values (numbers of certain coins) from relationships in the actual values (monetary worth of coins).
- Digit problems set a stage for study of properties of numbers, such as the principles behind casting out nine's, in number theory.

Customary Function Language

Discussion and Exercise

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- x is the <u>independent</u> variable; the x value in (x, y) is the <u>abscissa.</u>
  "Variable of choice" also means independent variable.
  Domain is the set of used or useful x values.
- y is the <u>dependent</u> variable; the y value in (x, y) is the <u>ordinate.</u>
  "Variable of consequence" also means dependent variable.
  Range is the set of used or useful y values.
- A function or equation is written so as to represent the dependent variable y (or $y = f(x)$ or $y = f(n)$) <u>in terms of</u> the independent variable x or (n). "Solving for y in terms of x" should be as familiar as "y is a function of x."

***"y* is a function of *x*"** is a very helpful phrase in many situations, as when a problem states that "Stock tank levels are a function of how much rain we get." Here, the stock tank levels are ____ (Which: x or y ?) "How much rain we get" is _______.

Then which variable is independent? ____ Dependent variable? ____

1. The squad paid $275 for the caps and sold them at $12 each. The relationship between the number n of caps sold and the profit $P = f(n)$ from the sale is represented by the function $P = f(n) = 12n - 275$. What is the dependent quantity in this functional relationship?

Note that the profit depends on the number of caps sold.

2. You rent a Toro chipper/shredder for a fee of $12, plus $15 per hour of use. Let c be the total cost and t be the number of hours of use. Write an equation that represents the dependent variable in terms of the independent variable.

A. $c = f(t) = \$15t + 12$
B. $t = f(c) = \$15c + 12$
C. $c = f(t) = \$12t + 15$
D. $t = f(c) = \$12c + 15$

Note here that <u>time</u> is very, very, often the independent variable that is graphed (therefore) on the x axis. It also belongs where we customarily see x in function notation such as choices A and C above.

3. Circle those ideas below that are associated with y:

abscissa ordinate domain range dependent variable

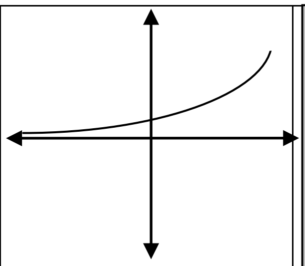
independent variable variable of choice variable of consequence

Pre-Calculus Parent Functions I

Translations on Exponential and Absolute Value Functions

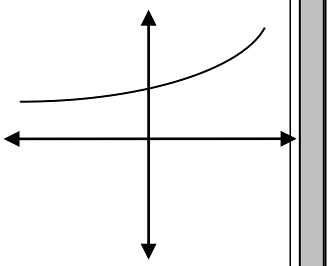
$y_1 = f(x) = 2^x$

x	y
–2	¼
–1	½
0	1
1	2
2	4



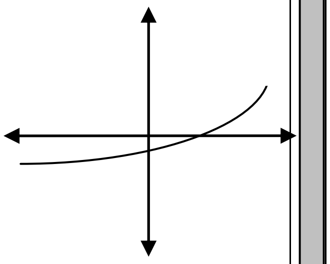
$y_2 = f(x) + 2 = 2 + 2^x$

x	y
–2	2¼
–1	2½
0	3
1	4
2	6



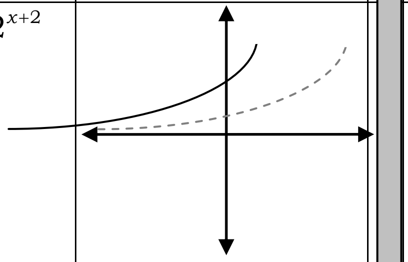
$y_3 = f(x) - 2 = {}^{-}2 + 2^x$

x	y
–2	–1.75
–1	–1.5
0	–1
1	0
2	2



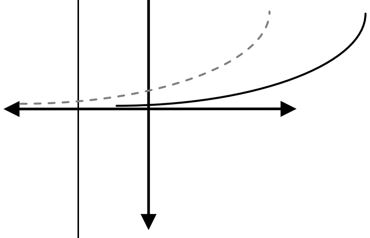
$y_4 = f(x+2) = 2^{x+2}$

x	y
–4	¼
–3	½
–2	1
–1	2
0	4



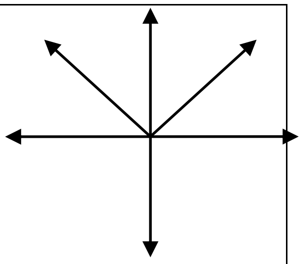
$y_5 = f(x-2) = 2^{x-2}$

x	y
0	¼
1	½
2	1
3	2
4	4



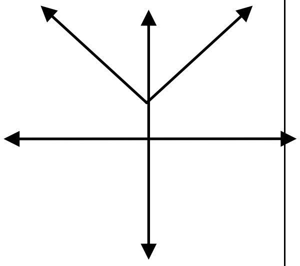
$y_1 = f(x) = |x|$

x	y
–2	2
–1	1
0	0
1	1
2	2



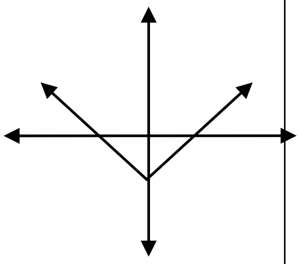
$y_2 = f(x) + 2 = |x| + 2$

x	y
–2	4
–1	3
0	2
1	3
2	4



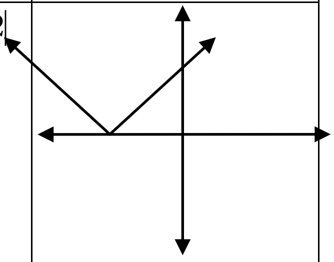
$y_3 = f(x) - 2 = |x| - 2$

x	y
–2	0
–1	–1
0	–2
1	–1
2	–2



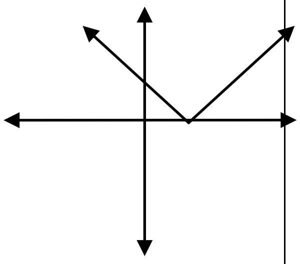
$y_4 = f(x+2) = |x+2|$

x	y
–4	2
–3	1
–2	0
–1	1
0	2



$y_5 = f(x-2) = |x-2|$

x	y
0	2
1	1
2	0
3	1
4	2

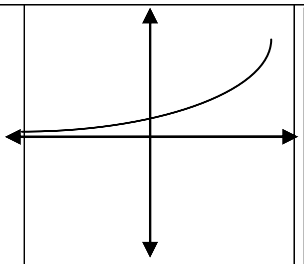


Pre-Calculus Parent Functions II

Reflections on Exponential and Square Root Functions

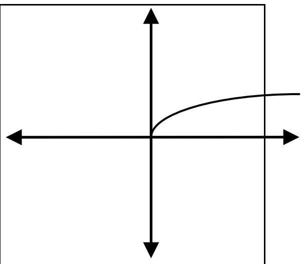
$y = f(x) = 2^x$

x	y
–2	¼
–1	½
0	1
1	2
2	4



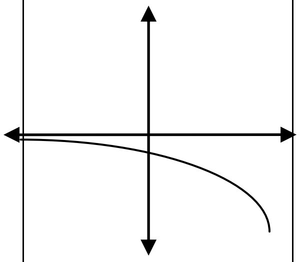
$y_1 = f(x) = \sqrt{x}$

x	y
0	0
¼	½
1	1
4	2
9	3



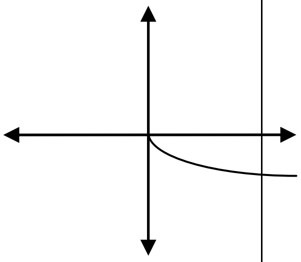
$y_6 = -f(x) = -2^x$

x	y
–2	–¼
–1	–½
0	–1
1	–2
2	–4



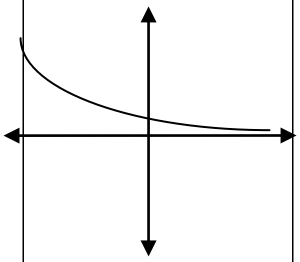
$y_2 = -f(x) = -\sqrt{x}$

x	y
0	0
¼	–½
1	–1
4	–2
9	–3



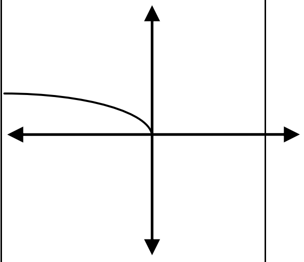
$y_7 = f(-x) = 2^{-x}$

x	y
–2	4
–1	2
0	1
1	½
2	¼



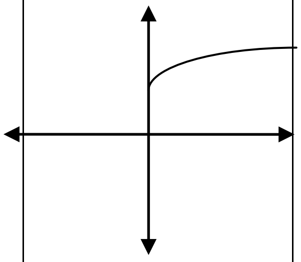
$y_3 = f(-x) = \sqrt{-x}$

x	y
0	0
–¼	½
–1	1
–4	2
–9	3



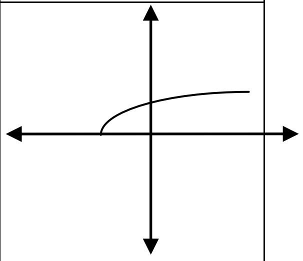
$y_4 = 2 + f(x) = 2 + \sqrt{x}$

x	y
0	2
¼	2½
1	3
4	4
9	5



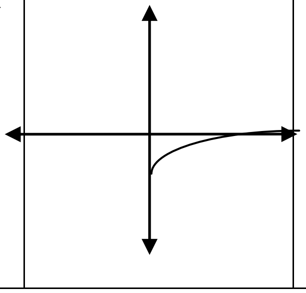
$y_5 = f(x+2) = \sqrt{x+2}$

x	y
–2	0
–1.75	½
–1	1
2	2
7	3



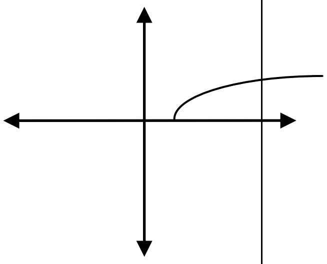
$y_6 = {}^{-}2 + f(x) = {}^{-}2 + \sqrt{x}$

x	y
0	–2
¼	–1.5
1	–1
4	0
9	1



$y_7 = f(x-2) = \sqrt{x-2}$

x	y
2	0
2¼	½
3	1
6	2
11	3

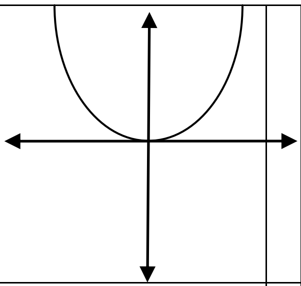


Pre-Calculus Parent Functions III

Translations Combined Parabola Vertices Revisited

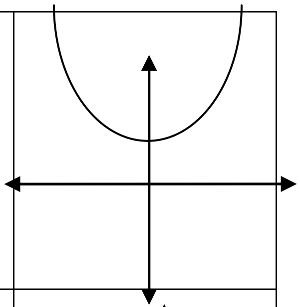
$y_1 = f(x) = x^2$

x	y
–2	4
–1	1
0	0
1	1
2	4



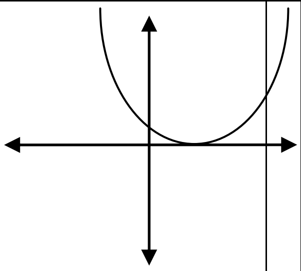
$y_3 = f(x) + 2 = x^2 + 2$

x	y
–2	6
–1	3
0	2
1	3
2	6



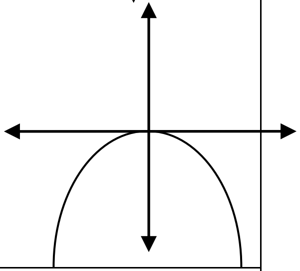
$y_2 = f(x-2) = (x-2)^2$

x	y
–2	2
–1	–1
0	–2
1	–1
2	2



$y_4 = -f(x) = -x^2$

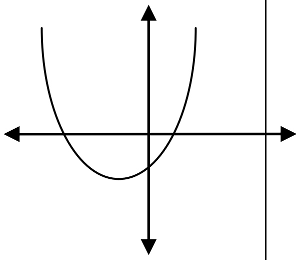
x	y
–2	–4
–1	–1
0	–0
1	–1
2	–4



$y_5 = f(x+1) - 2$

$= (x+1)^2 - 2$

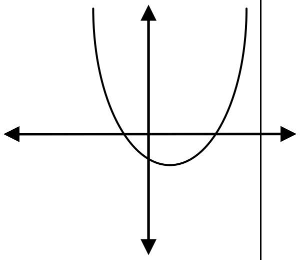
x	y
–3	2
–2	–1
–1	–2
0	–1
1	2



$y_6 = f(x-1) - 2$

$= (x-1)^2 - 2$

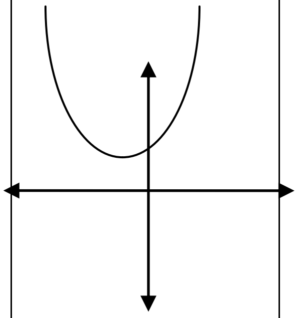
x	y
–1	2
0	–1
1	–2
2	–1
3	2



$y_7 = f(x+1) + 2$

$= (x+1)^2 + 2$

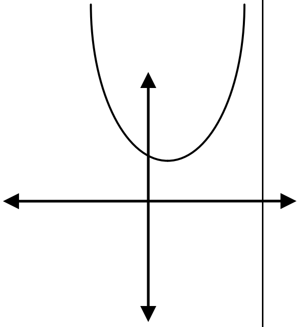
x	y
–3	6
–2	3
–1	2
0	3
1	6



$y_8 = f(x-1) + 2$

$= (x-1)^2 + 2$

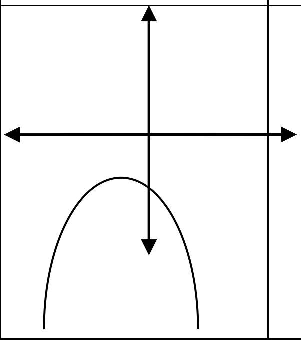
x	y
–1	6
0	3
1	2
2	3
3	6



$y_9 = -f(x+1) - 2$

$= -(x+1)^2 - 2$

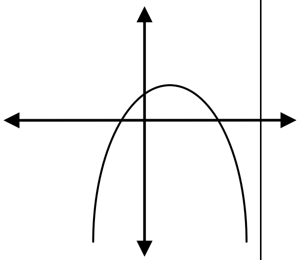
x	y
–3	–6
–2	–3
–1	–2
0	–3
1	–6



$y_0 = -f(x-1) + 2$

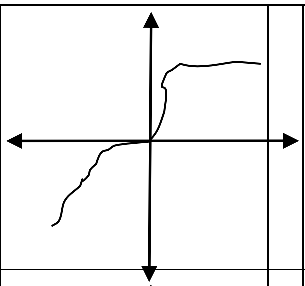
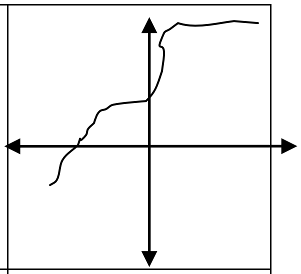
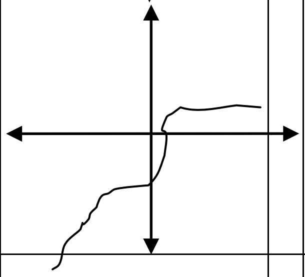
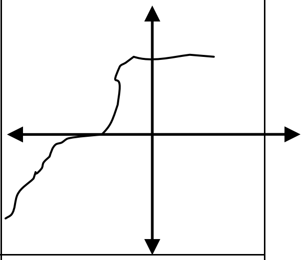
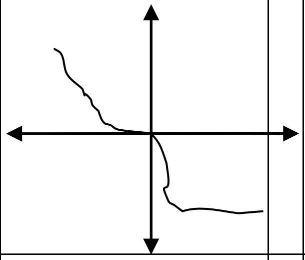
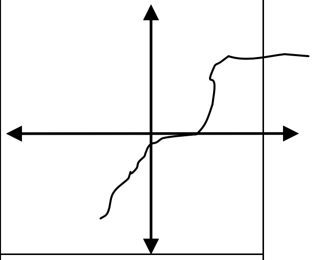
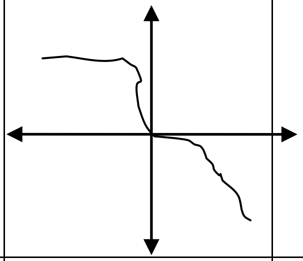
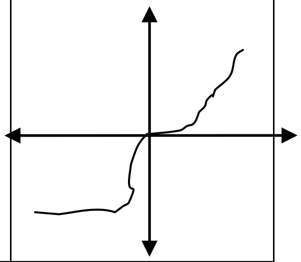
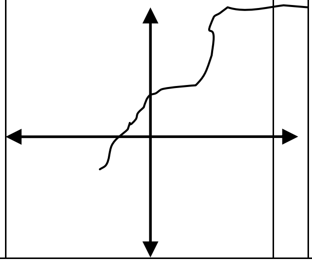
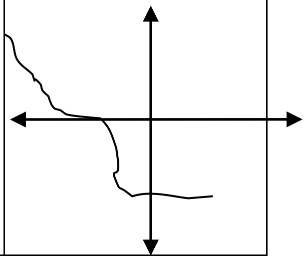
$= -(x-1)^2 + 2$

x	y
–1	–2
0	1
1	2
2	1
3	–2



Pre-Calculus Parent Functions IV

Translations Abstracted

$y_1 = f(x)$			$y_2 = f(x)+2$	
$y_3 = f(x)-2$			$y_4 = f(x+2)$	
$y_5 = -f(x)$			$y_6 = f(x-2)$	
$y_7 = f(-x)$			$y_8 = -f(-x)$	
$y_9 = f(x-2)+2$			$y_0 = -f(x+2)$	

<u>Transformations</u> are officially *translation* (“slide”), *reflection* (“flip”), and *rotation* (“spin”).

Graphs of the trigonometric functions (sine, cosine, tangent) will help introduce a fourth, *shear* (shrink/magnify).

Parent Functions and Transformations w/ examples

Following are systematic changes (transformations) that can be made to basic parent functions. Some of these affect *x*-values **before they are substituted** in the function, and these change the function in the *x*-direction. The *x*-direction transformations are

Shift left: $y = f(x+3)$ Shift right: $y = f(x-3)$

Compress to *y*-axis: $y = f(2x)$ Stretch from *y*-axis: $y = f(.5x)$

Flip or reflect left-to-right and right-to-left across the *y*-axis: $y = f(-x)$

Other transformations affect the *y*-values, **after the *x*-values had been substituted** in the function. These *y*-direction tranformations are

Shift up: $y = f(x)+3$ Shift down: $y = f(x)-4$

Stretch from *x*-axis: $y = 2f(x)$ Compress to *x*-axis: $y = .5f(x)$

Flip or reflect top-to-bottom and bottom-to-top over the *x*-axis: $y = -f(x)$

Examples using $y_1 = f(x) = 2^x = 2 \wedge x$ as the parent function.

$y_2 = f(x)-2 = 2^x - 2 = 2 \wedge x - 2$ 2 taken from ______ values. result:		$y_3 = f(x)+3 = 2^x + 3 = 2 \wedge x + 3$ 3 added to ______ values. result:
$y_4 = f(x-2) = 2^{x-2} = 2 \wedge (x-2)$ 2 taken from ______ values. result:		$y_5 = f(x+3) = 2^{x+3} = 2 \wedge (x+3)$ 3 added to ______ values. result:
$y_6 = -f(x) = -2^x = -2 \wedge x$ opposite of ______ values. result:		$y_7 = f(-x) = 2^{-x} = 2 \wedge (-x)$ opposite of ______ values. result:
$y_8 = f(3x) = 2^{3x} = 2 \wedge (3x)$ multiplying ______ values. result:		$y_9 = 3f(x) = 3\left(2^x\right) = 3(2 \wedge x)$ multiplying ______ values. result:

Odd and Even Functions Extravaganza

Stan Hartzler Archer City High School

	ODD	**EVEN**
	$y = f(x) = 3x^5 - 8x^3 + 3x$	$y = g(x) = x^4 - 8x^2 + 9$
1. What about these functions? (exponents?)		
2. Graph the functions.		
3A. What do you notice about the graphs? (S*ymmetry*?) 3B. The wire discussions?		
4. What is awesome about segments from a point on the graph?		
5. Complete charts. Pattern? Tie to graphs? *Note graph point pairs.* Substitute (by hand): $f(2)$, $f(-2)$, g(2), g(–2) $f(2) = 3 \bullet 2^5 - 8 \bullet 2^3 + 3 \bullet 2$ $f(-2) = 3(-2)^5 - 8(-2)^3 + 3(-2)$ $g(2) = 2^4 - 8(2)^2 + 9$ $g(-2) = (-2)^4 - 8(-2)^2 + 9$ Observation? Why?		
6A. Summarize odd/ even functions per parent-function transformations. 6B. Analyze.		
7. If (x, y) is on graph...		

Odd and Even Functions Extravaganza KEY?

<table>
<tr><th></th><th>ODD</th><th>EVEN</th></tr>
<tr><td></td><td>$y = f(x) = 3x^5 - 8x^3 + 3x$</td><td>$y = g(x) = x^4 - 8x^2 + 9$</td></tr>
<tr><td>1. What about these functions? (exponents?)</td><td>All exponents are odd.</td><td>All exponents are even.</td></tr>
<tr><td>2. Graph the functions.</td><td></td><td></td></tr>
<tr><td>3A. What do you notice about the graphs? (S<i>ymmetry</i>?)

3B. The wire discussions?</td><td><u>Symmetrical with respect to the origin.</u>
If graph for $x > 0$ were a bent wire, a 180° rotation about (0,0) would place wire atop graph for $x < 0$.</td><td><u>Symmetrical with respect to the <i>y</i> axis.</u>
If graph for $x > 0$ were a bent wire, a flip across $x = 0$ (<i>y</i>-axis) would place wire atop graph for $x < 0$.</td></tr>
<tr><td>4. What is awesome about segments from a point on the graph?</td><td>Segment from a graph point to origin has twin on other side.</td><td>Segment from a graph point to <i>y</i> axis ($\perp$) has twin on other side.</td></tr>
<tr><td>5. Complete charts. Pattern? Tie to graphs?
<i>Note graph point pairs.</i>
Substitute (by hand):
$f(2)$, $f(-2)$, g(2), g(−2)
$f(2) = 3 \bullet 2^5 - 8 \bullet 2^3 + 3 \bullet 2$
$f(-2) = 3(-2)^5 - 8(-2)^3 + 3(-2)$
$g(2) = 2^4 - 8(2)^2 + 9$
$g(-2) = (-2)^4 - 8(-2)^2 + 9$
Observation? Why?</td><td><table><tr><th><i>x</i></th><th><i>y</i></th></tr><tr><td>−3</td><td>$f(-3) = -522$</td></tr><tr><td>−2</td><td>$f(-2) = -38$</td></tr><tr><td>−1</td><td>$f(-1) = 2$</td></tr><tr><td>0</td><td>$f(0) = 0$</td></tr><tr><td>1</td><td>$f(1) = -2$</td></tr><tr><td>2</td><td>$f(2) = 38$</td></tr><tr><td>3</td><td>$f(3) = 522$</td></tr></table>Odd exponents change no input signs. $f(3)$ is opposite of $f(-3)$.
<i>Note graph point pairs.</i></td><td><table><tr><th><i>x</i></th><th><i>y</i></th></tr><tr><td>−3</td><td>18</td></tr><tr><td>−2</td><td>−7</td></tr><tr><td>−1</td><td>2</td></tr><tr><td>0</td><td>9</td></tr><tr><td>1</td><td>2</td></tr><tr><td>2</td><td>−7</td></tr><tr><td>3</td><td>18</td></tr></table>Even exponents turn negative input to positive results.
$f(-3) = f(3)$.
<i>Note graph point pairs.</i></td></tr>
<tr><td>6A. Summarize odd/even functions per parent-function transformations.

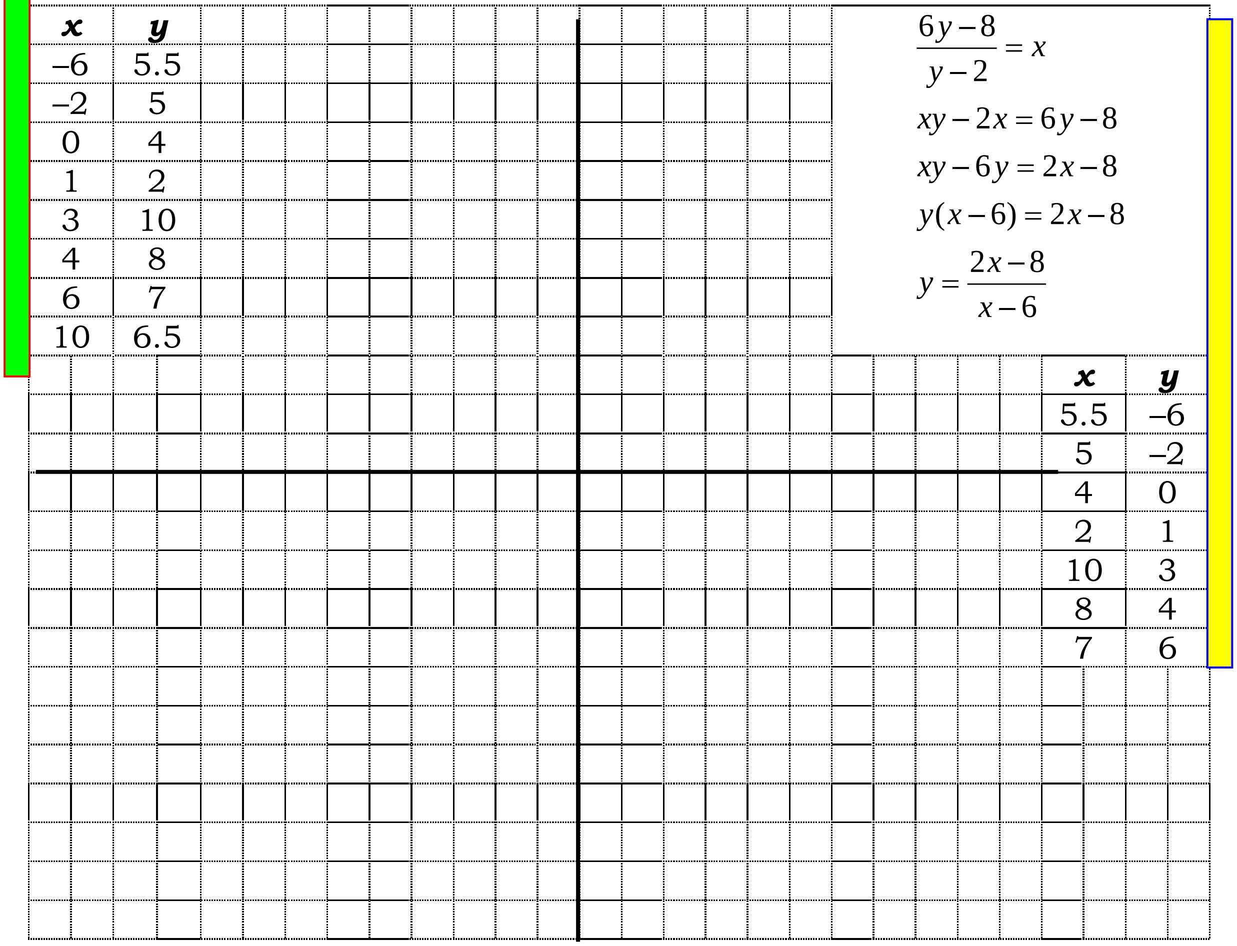
6B. Analyze.</td><td>$f(x) = -f(-x)$
<u><i>x</i> axis flip + <i>y</i> axis flip</u>
<i>means</i> <i>no change</i> in graph of $f(x)$.</td><td>$f(x) = f(-x)$
<u><i>y</i> axis flip</u>
<i>means</i> <i>no change</i> in graph of $f(x)$.</td></tr>
<tr><td>7. If (<i>x</i>,<i>y</i>) is on graph…</td><td>…then (−<i>x</i>, −<i>y</i>) is, too.</td><td>…then (−<i>x</i>, y) is, too.</td></tr>
</table>

Function Inverse Extravaganza II partial key

The idea that unites all embodiments of *function inverse* is?

$y = f(x) = \dfrac{6x-8}{x-2}$

Algebra search for $f^{-1}(x)$:



x	**y**
–6	5.5
–2	5
0	4
1	2
3	10
4	8
6	7
10	6.5

$$\frac{6y-8}{y-2} = x$$

$$xy - 2x = 6y - 8$$

$$xy - 6y = 2x - 8$$

$$y(x-6) = 2x - 8$$

$$y = \frac{2x-8}{x-6}$$

x	**y**
5.5	–6
5	–2
4	0
2	1
10	3
8	4
7	6

Composition:

Compose the charts:

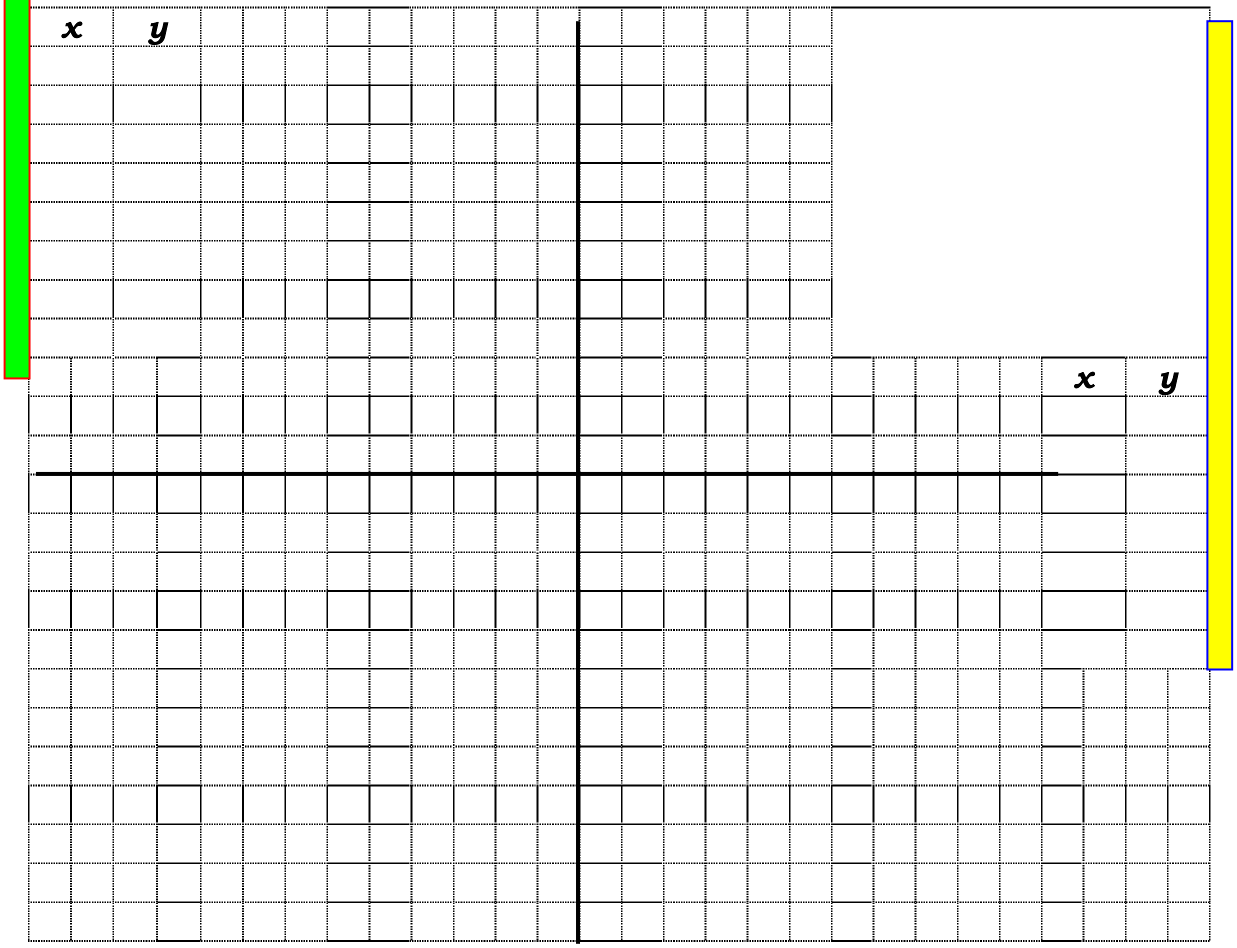
Compose the algebraic expressions:

Function Inverse Extravaganza II

The idea that unites all embodiments of *function inverse* is?

$y = f(x) = \dfrac{6x-8}{x-2}$

Algebra search for $f^{-1}(x)$:



Composition:

Compose the charts:

Compose the algebraic expressions:

Solving Extravaganza; Logarithm Equations

Dr. Stan Hartzler Archer City High School

A <u>variable term</u> contains an alphabet letter as a factor. In the first three equations below, the variable terms are $4x$, x, and $6x$.

To solve an equation, variable terms are collected, isolated, and unitized.

It is good to be mindful that

- *Solving by inspection is sometimes easiest.*
- *The goal is almost always to find <u>one</u> of the variable (unitize), after getting variable terms collected into just one term, and alone (isolated) on its side of the equal sign (<u>solitude</u>).*
- *The reverse operations are done to both sides of the equation.*
- *Planning is often facilitated by a trial substitution like* $x = 10$*, and then inverting the operations and reversing the order.*

For $4x = -7\frac{1}{2}$, the 4 must change to <u>*one*</u>. We anti-multiply, or divide, or multiply by the inverse (reciprocal); <u>we reverse or inverse operate</u>.

For $x + 5 = 3\frac{4}{7}$, <u>*one*</u> of the variable is there now. But the variable term has no <u>*solitude*</u>: 5 must become <u>*zero*</u>, leaving <u>*one*</u> *x* alone. To un-add, we subtract, or add the inverse (opposite); <u>we reverse or inverse operate</u>.

For $6x + 5 = -2\frac{3}{7}$, neither <u>*one*</u> of the variable nor solitude is in sight. Both un-multiplying and un-adding are needed. Which first? Either can be done, but the easiest solving process uses the critical principle.

Discovery and design of the easiest procedure can begin with pretending to solve be guesswork. <u>Pretend to guess that the variable *x* is a 10.</u>

Substitution (guesswork): only *think* order of operations	**Solving process**: *reverse order* <u>and</u> *invert operations*
1. Multiply 6	1. Subtract 5
2. Add 5	2. Divide 6

- If 10 is substituted for *x*, it is **first** multiplied by 6; **then** 5 is added.
- **Critical Principle:** The easiest <u>solving</u> process *reverses* (or *inverts*) *each operation* and <u>reverses the *order*</u> used for a substitution guess.
  The first step is un-adding the 5.
  The second step is un-multiplying the 6.

This use of inverse operations in contrary order might be described as unwrapping the package.

Other “basic” operations are subtraction and division: $\frac{x}{7} - 5 = .8$

More advanced operations include exponentiation. Four aspects will be mentioned here:

A. The variable term has an exponent:	(A) $(x+4)^3 = -27$
B. The exponentiation is disguised with a radical sign:	(B) $\sqrt{5-2x} = 3.2$ $(5-2x)^{\frac{1}{2}} = 3.2$
C. The variable term has a fraction exponent:	(C) $(x+4)^{\frac{2}{3}} = 9$
D. The variable or variable term is the exponent (an exponential equation):	(D) $243 = 96.2e^{-5t}$

In (A), the variable term is cubed. Unwrapping the package happens by taking the cube root of both sides, since the reverse or inverse of cubing is cube root. The complete solving plan again appears when something like 10 is substituted for x, and the order of operations is revealed and then(1) inverted and (2) reversed.

For (B), the square root (or the ½ power) is being taken of the variable term. The reverse or inverse of square root is squaring. The complete solving plan will have three steps this time.

For (C), the reverse of the $\frac{2}{3}$ exponent is a $\frac{3}{2}$ exponent, because

- exponents are multiplied in such an arrangement, and
- multiplying $\frac{3}{2} \bullet \frac{2}{3} = 1$, and 1 as an exponent disappears.

$$((x+4)^{\frac{2}{3}})^{\frac{3}{2}} = (9)^{\frac{3}{2}}$$

$$(x+4)^1 = (3^2)^{\frac{3}{2}}$$

etc.

The point of the lesson is now at hand. **Solve** $21 = 4^x$ for x.

How may an exponential equation like $21 = 4^x$ be solved for x? In other words, what exponent can we put on a 4 so that 21 is the result?

The answer for x has to be a number that is between 2 and 3, and much closer to 2. (Why?) But isn’t there a process to replace guesswork? Yes!

The process will require isolating the exponent x -- using a *logarithm* operation, a reverse or inverse of the exponent operation.

When the x in $21 = 4^x$ is isolated, it is called a logarithm (log for short). The language used in exponent isolation includes any of these:

- The exponent placed on a base of 4 to produce 21 is x.
- The logarithm with base 4 for 21 is x.
- The logarithm of 21, base 4, is x.
- Most briefly, $\log_4 21 = x$.

The Briggs (or *common*) system uses 10 as a base. So "log 3" means "$\log_{10} 3$". When no base for the logarithm is given, 10 is a *default* base.

Actually finding out what x is in $21 = 4^x$ requires a theorem. Try finding $\log_4 21 = \dfrac{\log 21}{\log 4} =$ ________ The result should exceed 2 a bit, as predicted.

Again, the reverse or inverse of exponent is logarithm, just as

addition and subtraction are inverses of each other
multiplication and division are inverses of each other
squaring and square root are inverses of each other, etc.

The **log** button on a calculator is short for logarithm for base 10.

For (D) above: for exponentiation, reversal is expressed as a logarithm, which is an exponent (especially an isolated exponent). Recall the basic form change is $a = b^c \Leftrightarrow \log_b a = c$.

- It may be said that "A logarithm c is the exponent placed on the base b to produce a." So both members of $\log_b a = c$ are exponents.

Substitution (guesswork): only *think* order of operations For the example (D) $243 = 96.2e^{-5t}$, we *think* (guessing) about t = 10.	× –5 exponentiate on e × 96.2

From this thinking, a **solving process** develops, reversing order and inverting operations.

1. Divide by 96.2.
2. Log both sides, base e ($\log_e = \ln$) -- form change.
3. Divide by −5.

$$t = \frac{\log_e(\frac{243}{96.2})}{-5}$$

Solve $1.7 - 4\log_6 \dfrac{x-2}{5} = 9.7$. Think guesswork to develop a process plan.

Connection: Distance/Area/Volume

Dr. Stan Hartzler Archer City High School

Distance (length): number of segments* needed to connect two points.

Perimeter: distance around a region

Area: number of squares* needed to cover a region or surface.

Surface Area: sum of face areas for a solid

Volume: number of cubes* needed to fill a space.

***of uniform size**

Arc Length and Sector Area Connected

For a circle with radius 12, do the circle blanks.

For a sector of that circle made by a central angle of 200°,

Arc Length (length of crust of apple pie slice)	**Sector Area** (Number of covering Hershey squares)
$\frac{200^\circ}{360^\circ}$ × full perimeter	$\frac{200^\circ}{360^\circ}$ × full area
equals what?	equals what?

Pythagorean Triples per Sierpinski page 8 and Proof of Right Triangle Area = *xy*

"Every primitive Pythagorean triple (a, b, c) where b is an even integer is obtained only once per the following, where u and v are odd and $u > v$:

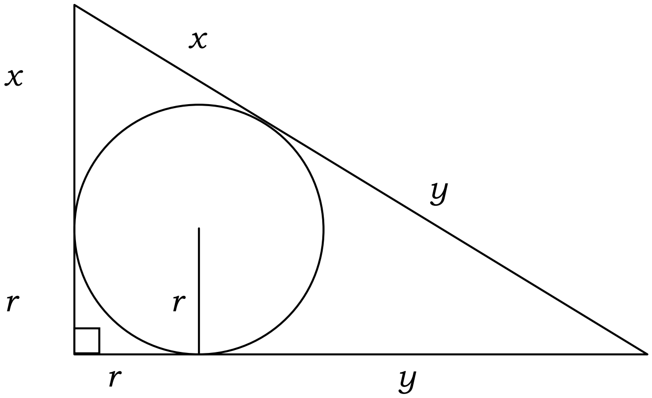
$$a = uv \qquad b = \frac{u^2 - v^2}{2} \qquad c = \frac{u^2 + v^2}{2}$$

Here, u and v represent all pairs of odd, relatively prime natural numbers." -- Waclaw Sierpinski, *Pythagorean Triangles, The Scripta Mathematics Studies Number Nine.* New York City: Graduate School of Science, Yeshiva University, 1962.

****

Proof of Right Triangle Area = *xy*

Dr. Stan Hartzler Archer City High School



Area = $\frac{1}{2} r \bullet$ perimeter = $\frac{1}{2} r(2r + 2x + 2y) = r(r + x + y)$

$$(x+r)^2 + (y+r)^2 = (x+y)^2$$
$$x^2 + 2xr + r^2 + y^2 + 2yr + r^2 = x^2 + 2xy + y^2$$
$$2xr + r^2 + 2yr + r^2 = 2xy$$
$$2xr + 2r^2 + 2yr = 2xy$$
$$xr + r^2 + yr = xy$$
$$r(x + r + y) = xy = \text{AREA}$$

Fun Relationships

$$r=\frac{a+b-c}{2}=\frac{v(u-v)}{2}=\frac{\sqrt{x^2+6xy+y^2}-(x+y)}{2}$$

$$x=a-r=\frac{v^2+uv}{2}$$
$$y=b-r=\frac{u^2-uv}{2}$$

$$a=x+r=\frac{\sqrt{x^2+6xy+y^2}+x-y}{2}$$
$$b=y+r=\frac{\sqrt{x^2+6xy+y^2}-x+y}{2}$$
$$c=x+y$$

Perimeter = $a+b+c=2(r+x+y)=x+y+\sqrt{x^2+6xy+y^2}=uv+u^2$

Area = $\dfrac{ab}{2}=xy=\dfrac{u^3v-uv^3}{4}$

Developing *u*, *v* for a given *r*

Example: $r = 3 \times 5 \times 7 = 105$

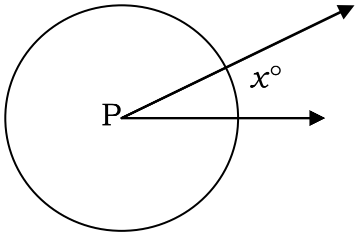
$$r=\frac{uv-v^2}{2}\Rightarrow u=\frac{v^2+2r}{v}=\frac{v^2+210}{v}$$

v	u	a	b	c
1	211	211	22260	22261
3	73	219	2660	2669
5	47	235	1092	1117
7	37	259	660	709
15	29	435	308	533
21	31	651	260	701
35	41	1435	228	1453
105	107	11235	212	11237

The number of distinct values of (u, v) for a given r of n odd prime factors is 2^n. And note that $u_k+v_k=u_{2^n+1-k}+v_{2^n+1-k}$.

Angle Measures in Circles lesson 67

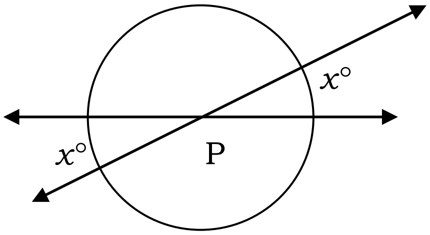
Line/Segment Position and *Measure* with Illustration



$m\angle P = x°$

Central angle

Central angles from lines

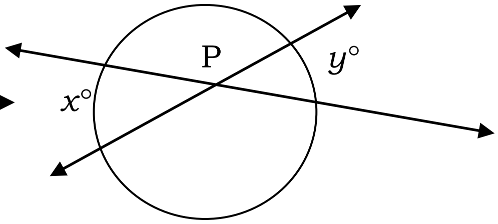


$m\angle P = x°$

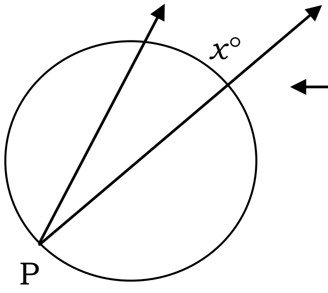
Angle equals arc

Vertex is moved off of center but is still in interior.

$$m\angle P = \frac{x+y}{2}$$



Angles equal average of intercepted arcs



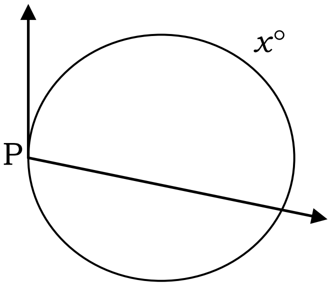
$$m\angle P = \frac{1}{2}x°$$

Vertex is ***on*** circle

Neither ray tangent

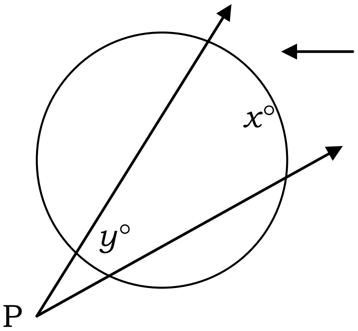
One ray tangent

Angle is half of intercepted arc.



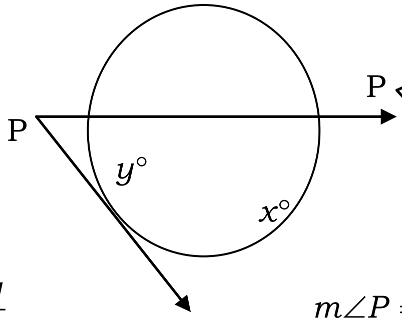
$$m\angle P = \frac{1}{2}x°$$

Vertices are ***outside*** circle

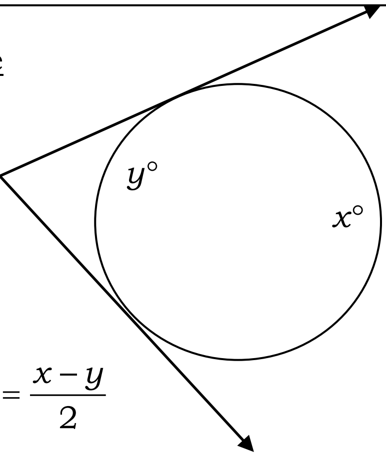


$$m\angle P = \frac{x-y}{2}$$

Neither ray tangent



One ray tangent



$$m\angle P = \frac{x-y}{2}$$

Both rays tangent

Angle is one-half of difference of intercepted arcs.

Ratios in Similar Solids

Dr. Stan Hartzler Archer City High School

One-dimensional measure refers to distance or length (number of segments). Other manifestations or embodiments of distance are height, perimeter, altitude, length of diagonals, edge, etc.

Two-dimensional measure refers to area (number of squares). Manifestations of this for working with solids include painting outside surfaces, or wall-papering, carpeting, tiling, and so on.

Three-dimensional measure refers to volume (number of cubes). Manifestations include filling the space inside a solid with a liquid or sand.

The Main Story

The ratio of corresponding one-dimensional measures of two similar solids is $\frac{a}{b}$ when…

…the ratio of corresponding two-dimensional measures of two similar solids is $\frac{a^2}{b^2}$, and when…

…the ratio of corresponding three-dimensional measures of two similar solids is $\frac{a^3}{b^3}$.

Example:

The ratio of corresponding one-dimensional measures of two similar pyramids is $\frac{4}{5}$ when…

…the ratio of corresponding two-dimensional measures of two similar pyramids is $\frac{4^2}{5^2} = \frac{16}{25}$, and when…

…the ratio of corresponding three-dimensional measures of two similar pyramids is $\frac{4^3}{5^3} = \frac{64}{125}$.

More typically, when the ratio of the altitudes of two similar pyramids is $\frac{4}{5}$, then the ratio of the areas of two corresponding sides is $\frac{4^2}{5^2} = \frac{16}{25}$, and the ratio of their volumes is $\frac{4^3}{5^3} = \frac{64}{125}$.

The Dilbert Head

A Study of a Partially Disassembled Cylinder

Walls of a prism or ______________ rise from a base to an identical

______________________________.

Walls of a pyramid or _________ rise from a base to meet at a

_______________________________.

1. The area of these walls is called the ____________________ _____________________ ___________.

2. The sum of the ________________ and the above wall area is called the

__________________ ___________________ _____________.

3. Below is a partially disassembled _____________________.

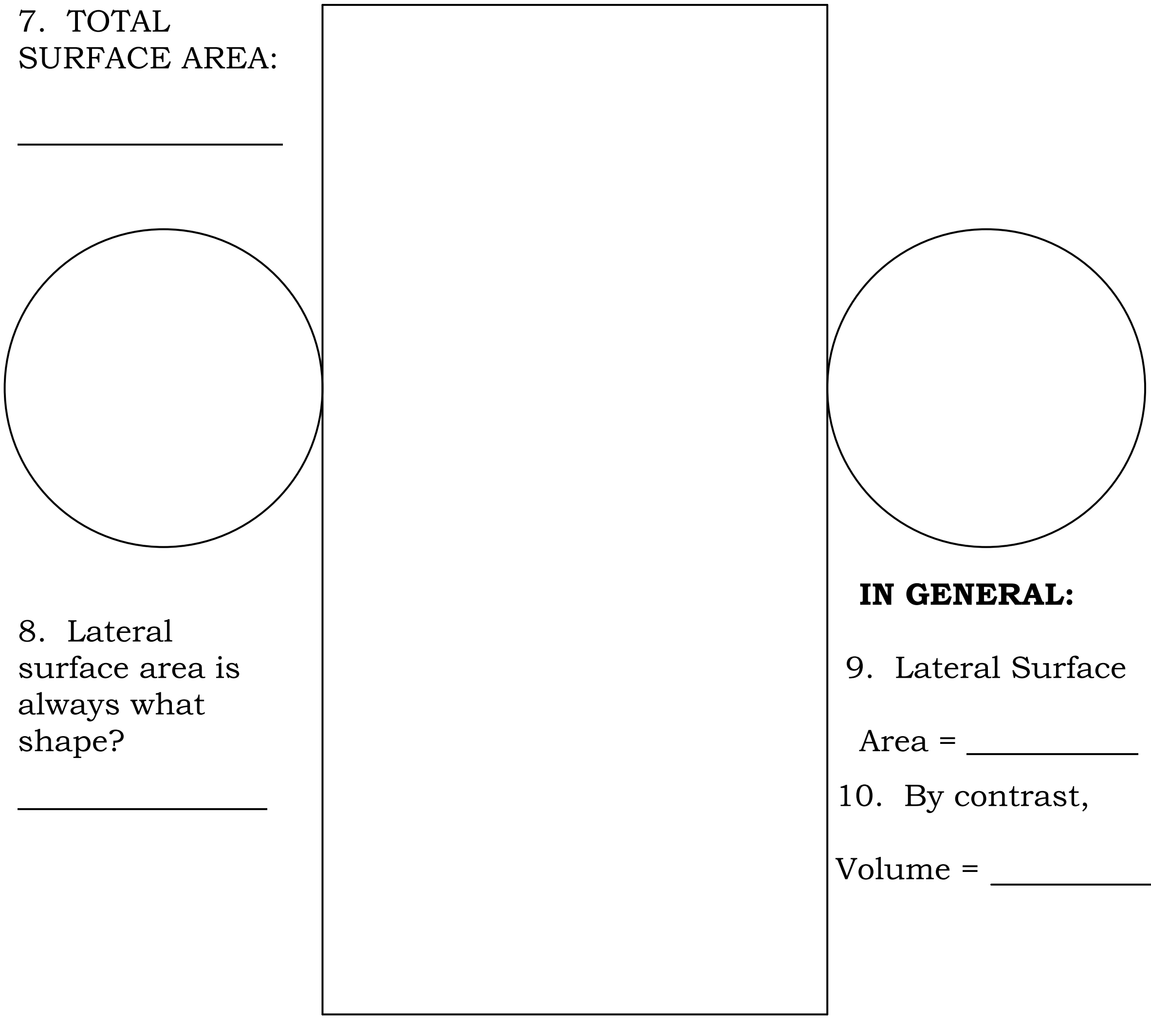
4. The three shapes are ____

5. The diameter of each ear is 43 mm. Find the area. _________________

6. Dilbert's head's width is 68 mm. What idea gives the length (top to bottom) of Dilbert's Head? ____________________ Now find the head area.

7. TOTAL SURFACE AREA:

____



8. Lateral surface area is always what shape?

____

IN GENERAL:

9. Lateral Surface

Area = ___________

10. By contrast,

Volume = _________________

Dilbert Head II

A Study of a Partially Disassembled Prism

9. Walls of a prism or _____________ rise from a base to an identical

_____________________________.

10. The area of these
walls is called the ___________________ ____________________ ___________.

11. The sum of the _______________ and the above wall area is called the

_________________ __________________ ____________.

12. Below is a partially disassembled ____________________.

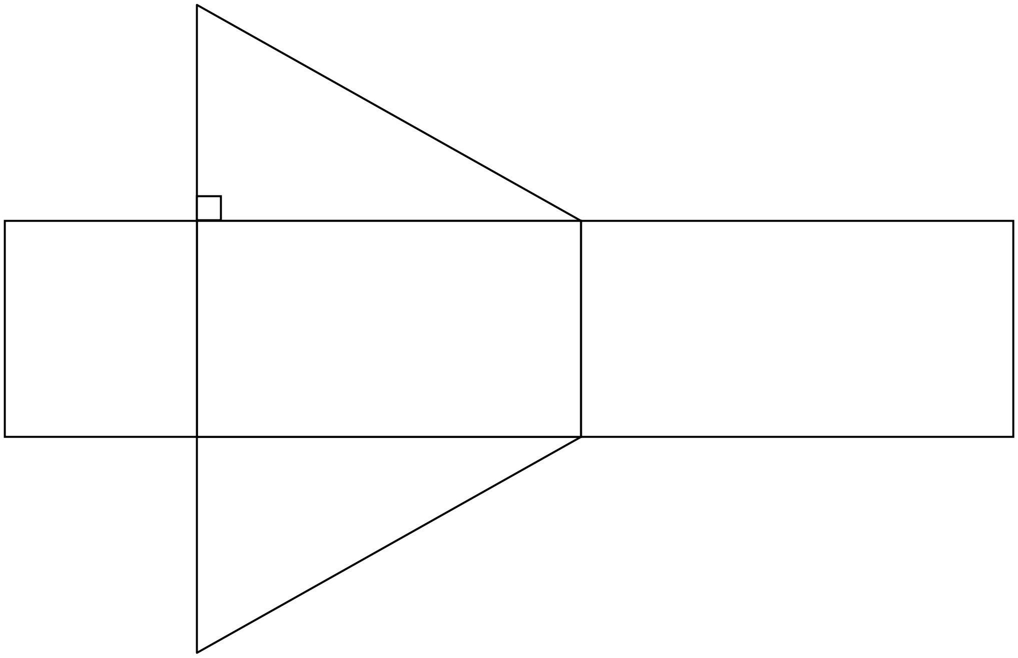
13. The three shapes are ____

14. The edges of each ear are 8, 15, and the longest = __________. Find the area of each ear. ________________

15. Dilbert's head's width is 68 mm. What idea gives the length (top to bottom) of Dilbert's Head? ___________________ Now find the head area.

16. TOTAL SURFACE AREA: ________________

17. Lateral surface area is always what shape? _______________



Mutation of the Dilbert Head

A Study of a Partially Disassembled Square Prism

Walls of a ____________ or ____________ rise from a base to an identical ___________________.

Walls of a ______________ or __________ rise from a base to meet at a ______________________.

1. Area of these walls is _______________ __________________ _________.

2. The sum of the _______________ and the above wall area is called the

_________________ __________________ ____________.

3. Below is a partially disassembled ____________________.

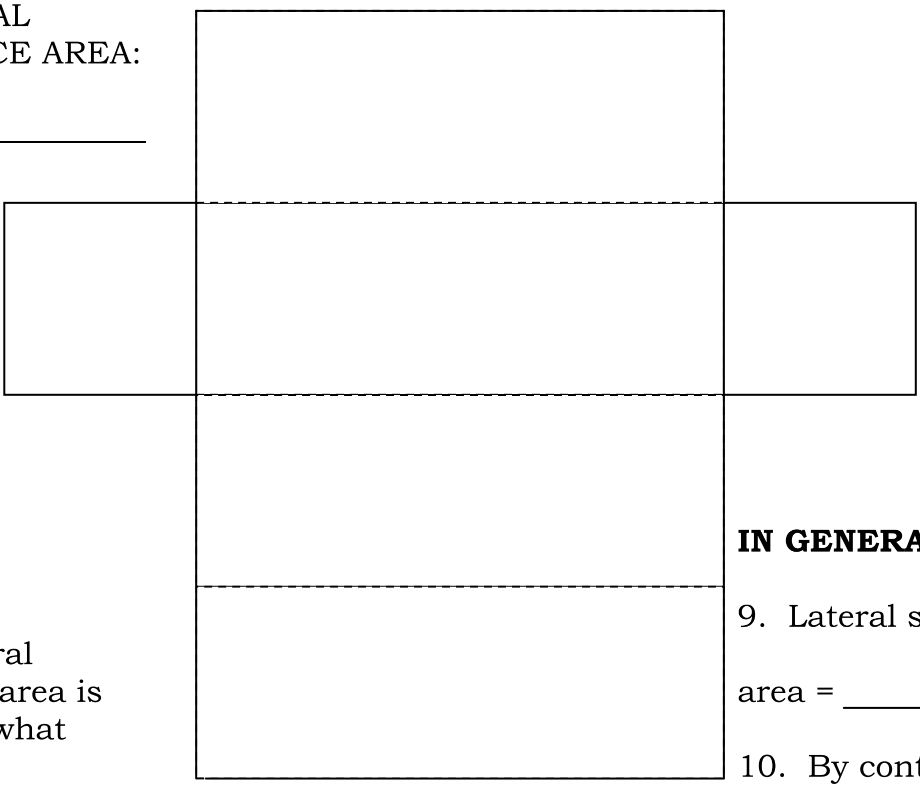
4. The three shapes are ____

5. The width of each square ear is 2.5 cm. Find the area. _____________

6. Dilbert's head's width is 7 cm. What idea gives the length (top to bottom) of Dilbert's Head? ___________________ Now find the head area. _____________

7. TOTAL
SURFACE AREA:

____



8. Lateral
surface area is
always what
shape?

____

IN GENERAL:

9. Lateral surface

area = _________________

10. By contrast:

volume = _______________

The Shrunken Dilbert Head

A Study of a Partially Disassembled Cylinder

11. Below is a partially disassembled ____________________.

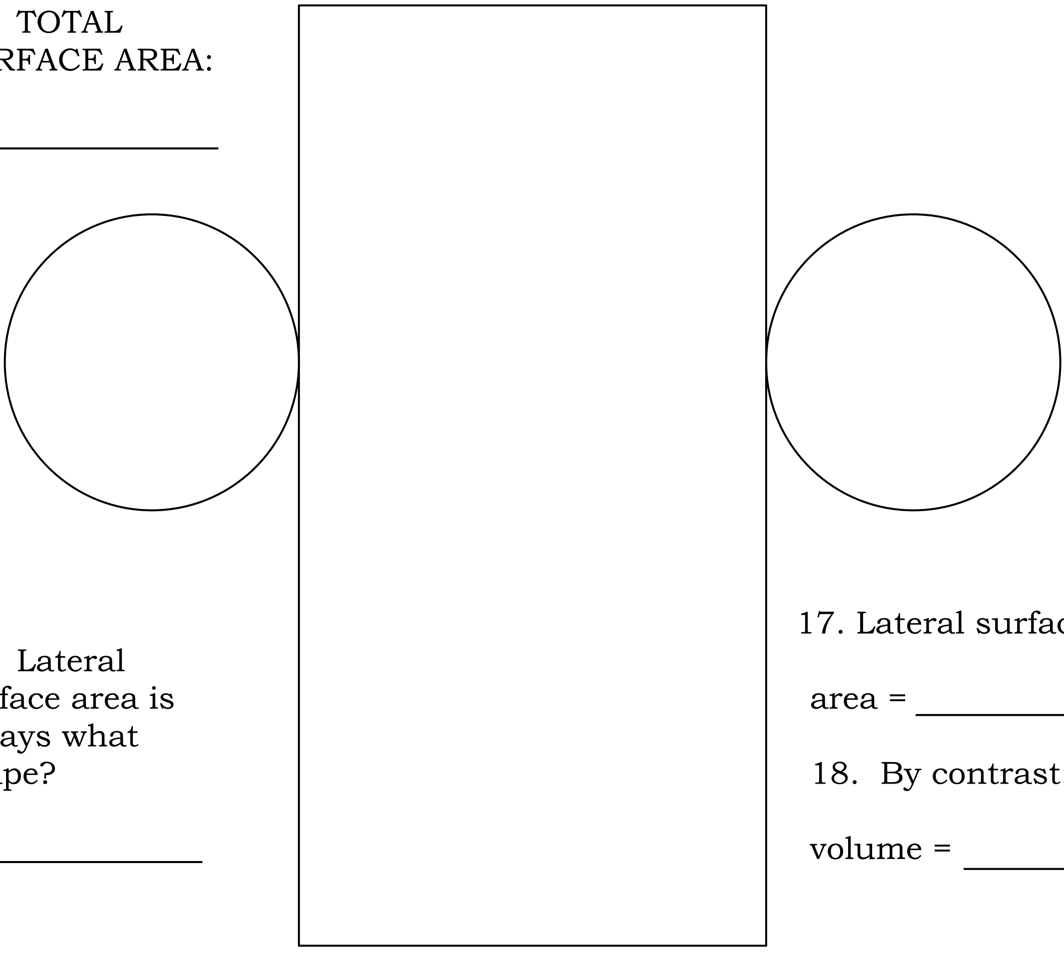
12. The three shapes are ____

13. The diameter of each ear is 40 mm. Find the area. ________________

14. Dilbert's head's width is 60 mm. What idea gives the length (top to bottom) of Dilbert's Head? ___________________ Now find the head area. _____________

15. TOTAL
SURFACE AREA:

____



16. Lateral
surface area is
always what
shape?

____

17. Lateral surface

area = _________________

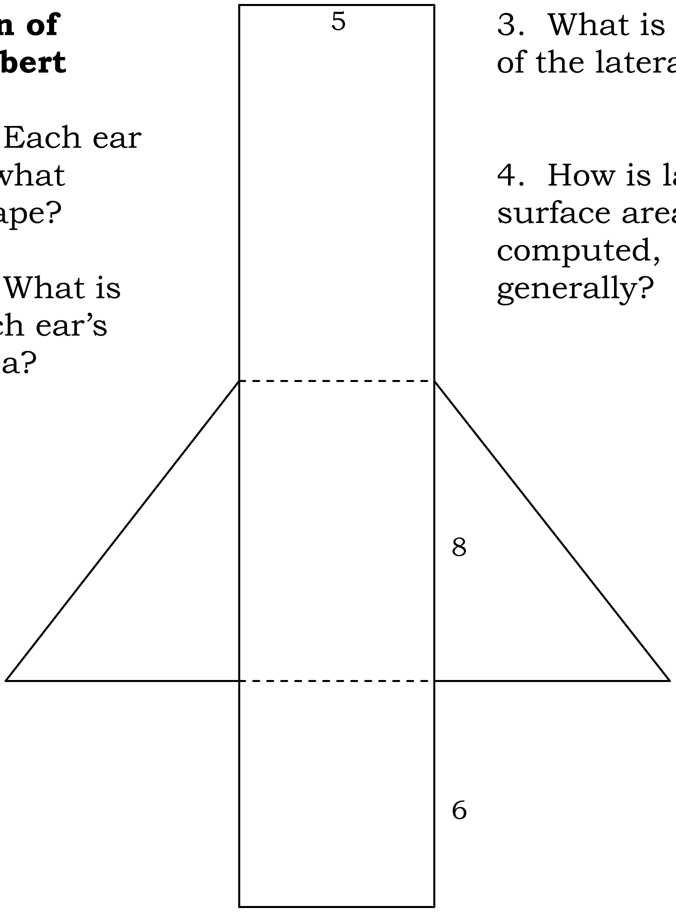
18. By contrast:

volume = ________________

Son of Dilbert

1. Each ear is what shape?

2. What is each ear's area?



3. What is the shape of the lateral surface?

4. How is lateral surface area computed, generally?

5. What is the lateral surface area?

6. What is the total surface area?

7. How is volume computed, generally?

8. What is the name of the solid figure formed by the "net" or Dilbert Head?

9. What is the volume of the solid figure formed by the net?

Deductive Structure in Geometry and...

Dt. Stan Hartzler Archer City High School

Veterans of decent geometry courses will recall some of the following elements, and perhaps some of the structure as well.

Undefined Terms:
- point
- line
- plane
- space ... between (?)...

Definitions (from undefined terms):
- ray, segment, polygon,...
- parallel, perpendicular, skew,...
- acute, right, obtuse...
- scalene, isosceles, equilateral,...

Unproved Hypotheses (built on experience with undefined and defined ideas):
- Parallel Postulate
- SSS, SAS, ASA $\Delta \cong$
- reflexive, symmetric, transitive
- corresponding angles are $\cong$.
- AA $\Delta \sim$
- ... and *uncountably* many more.

Theorems (built from terms, hypotheses, and other theorems):
- Vertical angles are congruent
- All right angles are congruent.
- AAS $\Delta \cong$
- Pythagorean Theorem
- Alternate interior angles are $\cong$.
- $\sum$ of Δ $\angle$ measures = 180°
- ... and *uncountably* many more.

In geometry, attention is given to the notion that *all postulates would dearly love to become theorems*. There is a weird status system working here, even stranger that the status systems that exist among people.

The effort to turn the Parallel Postulate into a theorem led to other kinds of geometry, the *non-Euclidean geometries*, discussed briefly in ACISD geometry classes.

Attention should be given to the issue of the *uncountable* collection of Euclidean geometry postulates. In an effort to gain some strange status for humanity, many mathematicians have tried to show that mathematics is entirely a human invention.

These efforts failed. A 20th-Century German mathematician, Gödel, showed that for any mathematics system, including arithmetic, complete knowledge cannot be obtained deductively with a countable list of assumptions (postulates). Gödel's proof of this idea is rather simple, and gave him rock-star-plus status among great people of the 20th Century.

The countable issue is a fun issue. The set of **rational numbers**, while infinite, is a countably infinite set. The set of **irrational numbers** is uncountably infinite. So is the set of geometry postulates.

For most mathematicians, this shoots down the idea that truth in mathematics comes entirely from people. This raises a question: *"Then where does it come from?"* Some status-seekers are very troubled by this question. Your teacher is not troubled at all -- in fact, delighted -- by that question. DONE.

Connection: Form and Number Definitions

Dr. Stan Hartzler Archer City High School

1		**Form**	**is to**	**Number**
2	as	$\cong$	**is to**	$=$
3	as	$\overline{AB}$	**is to**	$\boldsymbol{AB}$
4	as	$\angle DFG$	**is to**	$m\angle DFG$
5	as	complementary, supplementary: “merging forms to right angle or straight angle”	**is to**	complementary, supplementary: “measures add to 90° and 180°”
6	as	perpendicular lines: “lines meeting to form adjacent congruent angles (Euclid)”	**is to**	perpendicular lines: “lines meeting to form 90° angles or angles of equal measure”
7	as	straight angle: “union of opposite rays with a common vertex”	**is to**	straight angle: “angle with measure of 180°”
8	as	diameter: “chord containing the center”	**is to**	diameter: “ chord with measure = 2r”
9	as	parallel lines (in a plane) and parallel planes: “never intersecting”	**is to**	parallel lines and planes: “everywhere equidistant”
10	as	MEDIAN CONSTRUCTION	**is to**	EQUATION OF MEDIAN
11	as	ALTITUDE CONSTRUCTION	**is to**	LENGTH/EQUATION OF ALTITUDE
12	as	PARALLEL LINES BY CONSTRUCTION	**is to**	PARALLEL LINES BY EQUAL SLOPES
13	as	PERPENDICULAR LINES BY CONSTRUCTION	**is to**	PERPENDICULAR LINES BY SLOPE RELATIONSHIP

14	as	“TAPESTRY” ACTIVITY COLUMN	**is to**	“TAPESTRY” ANALYSIS COLUMN
15	as	?	**is to**	?

Discussion

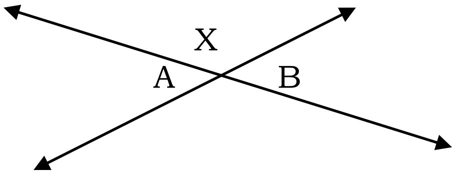
The original idea for the page-one chart is a “sound-bite” definition of mathematics:

Mathematics is the study of the invariant (unchanging) aspects of number and form.

Geometry lends itself strongly to study of both number and form.

The proof aspect of a geometry course, for example, allows for use of both the number definition of an idea and the form definition of the same idea to justify steps in the same two-column proof. An example follows.

Theorem: Vertical angles are congruent.



Given: Vertical angles A and B as shown.

Prove: $\angle A \cong \angle B$

Statement	**Reasons**
1. Vertical angles A and B as shown.	1. Given
2. $\angle A, \angle X$ are supplementary; $\angle B, \angle X$ are supplementary	2. “Supplementary” (form)
3. $m\angle A + m\angle X = 180°$; $m\angle B + m\angle X = 180°$	3. “Supplementary” (number)
4. $m\angle A + m\angle X = m\angle B + m\angle X$ $-m\angle X \qquad -m\angle X$	4. Transitive
5. $m\angle A = m\angle B$	5. Subtraction
6. $\angle A \cong \angle B$	6. “$\cong$”

Geometry Proof Strategies

Dr. Stan Hartzler   Archer City High School

The list that follows grows (dynamic development) as the geometry course progresses through the year. In this writer's classroom, it is on the whiteboard. Any time that the class is stuck for ideas, this teacher arises from the overhead and stands by the list, reviewing aloud while pointing to what is spoken. If needed, class attention is directed alternately from the proof efforts to the list, and back to the list again. The light goes on eventually. Meanwhile, a good review of the course has taken place, in part because of teacher embellishment on bad guesses.

A. Look for congruent pieces of figures, and mark with tic marks:
- from GIVEN
- from recall
    - vertical angles
    - right angles
    - isosceles triangle or trapezoid
    - equal measures
    - reflexive property
    - transitive property

B. Look for congruent triangles or similar triangles.
- triangle congruence postulates
- triangle similarity postulates

C. Use CPCTC, and then revisit part A above
- In similarity, use Corresponding Angles in Similar Triangles ≅
- …or Corresponding Sides of Similar Triangles are Proportional

Out of clutter, find simplicity.

-- Albert Einstein, Three rules of work.

Geometric Representation of Means

after Dr. Titu Andrescu

Given two quantities *a* and *b* represented by segments of these respective lengths, the means can be represented and compared geometrically. The means are identified as follows:

- arithmetic mean AM = $\frac{a+b}{2}$
- geometric mean GM = $\sqrt{ab}$
- harmonic mean HM = $\frac{2ab}{a+b}$
- root mean square RMS = $\sqrt{\frac{a^2+b^2}{2}} = \frac{\sqrt{a^2+b^2}}{\sqrt{2}}$

Sketchpad ® steps follow:

1. Draw diameter $\overline{AB}$.
2. Construct midpoint at C
3. Construct circle, center at C and radius controller at B.
4. Locate D on $\overline{AB}$, not at C.
5. Construct $\perp$ at C.
6. Construct intersection point of circle and $\perp$ at E.
7. Construct $\overline{CE}$ and hide perpendicular line. CE = AM
8. Construct perpendicular at D.
9. Construct intersection point of circle and $\perp$ at H.
10. Construct $\overline{DH}$ and hide perpendicular line. DH = GM, as ΔAHB is a right triangle.
11. Copy $\overline{AD}$ and rotate it $90°$ about center *A*.
12. Copy $\overline{BD}$ and rotate it $270°$ about center *B*.
13. Construct “guy wire” segments $\overline{AQ}$ & $\overline{PB}$ and intersection point F. Why

this is on $\overline{DH}$ is a mystery at present. The length of $\overline{DF}$ is half the HM as in the Two Pole problem.

14. Copy $\overline{DF}$ and rotate the copy $180°$ with center F. $\overline{GD}$ = HM
15. Copy $\overline{DB}$ and rotate the copy $270°$ with center D. New end is L.
16. Construct $\overline{AL}$. AL = $\sqrt{a^2+b^2}$
17. Construct midpoint of $\overline{AL}$. Select the midpoint as a center of rotation.
19. Copy $\overline{AL}$ and rotate it $90°$ though center. New endpoints are P and J.

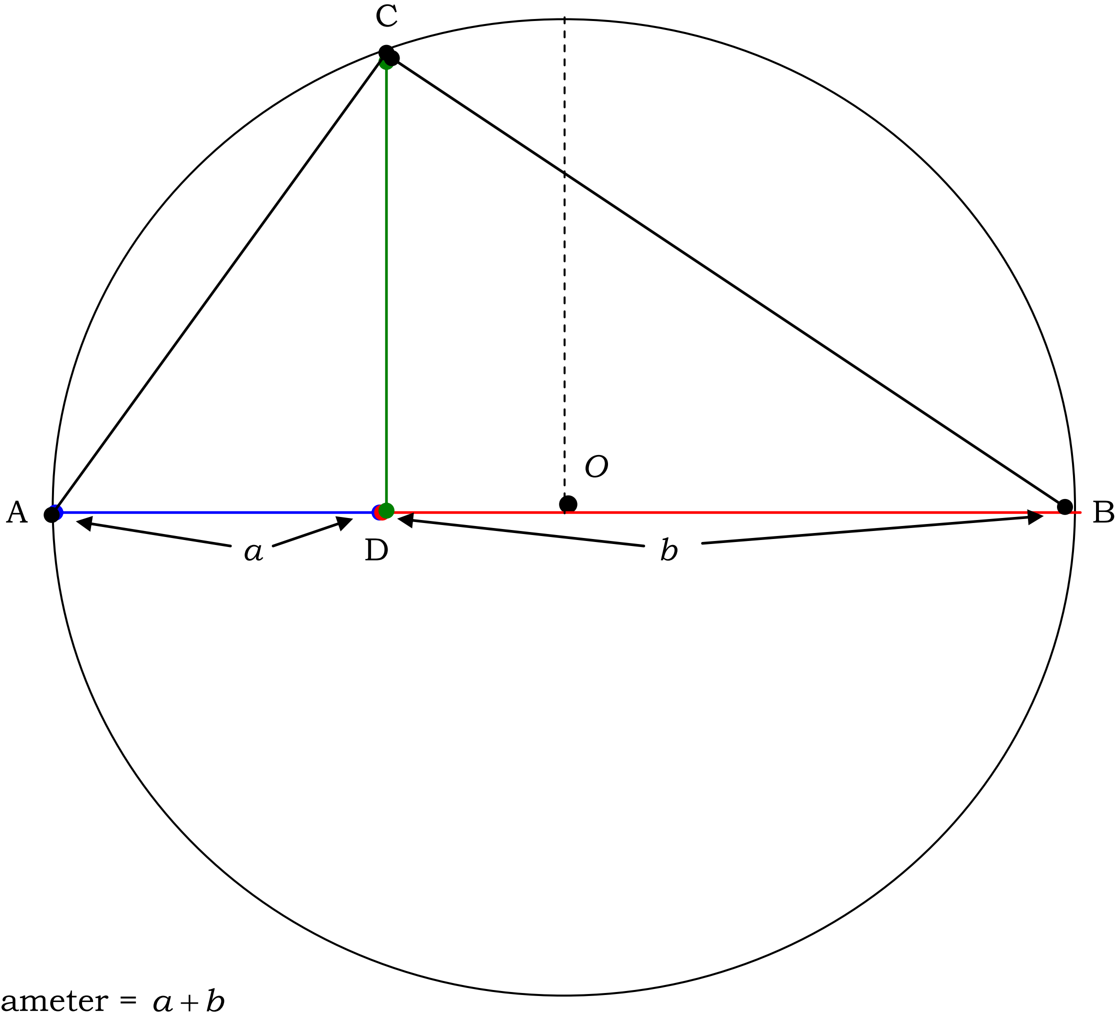
Why J is on circle at end of radius from D is a mystery at present.

20. $DJ = JM = MP = PD =$ $\sqrt{\frac{a^2+b^2}{2}} = \frac{\sqrt{a^2+b^2}}{\sqrt{2}}$ = RMS, longer than AM radius.

Conclusion: RMS $\geq$ AM $\geq$ GM $\geq$ HM .

Geometric Representation of Geometric Mean

With ΔABC having $\overline{AB}$ as a diameter of $\odot O$, $\angle ACB$ is a right angle and $\overline{CD}$ is an altitude to hypotenuse $\overline{AB}$. With similar triangles and $\frac{\text{short leg}}{\text{long leg}}$ scheme, $\frac{AD}{DC} = \frac{DC}{CB}$, and CD is **GM** of AD and DB.



Diameter = $a + b$

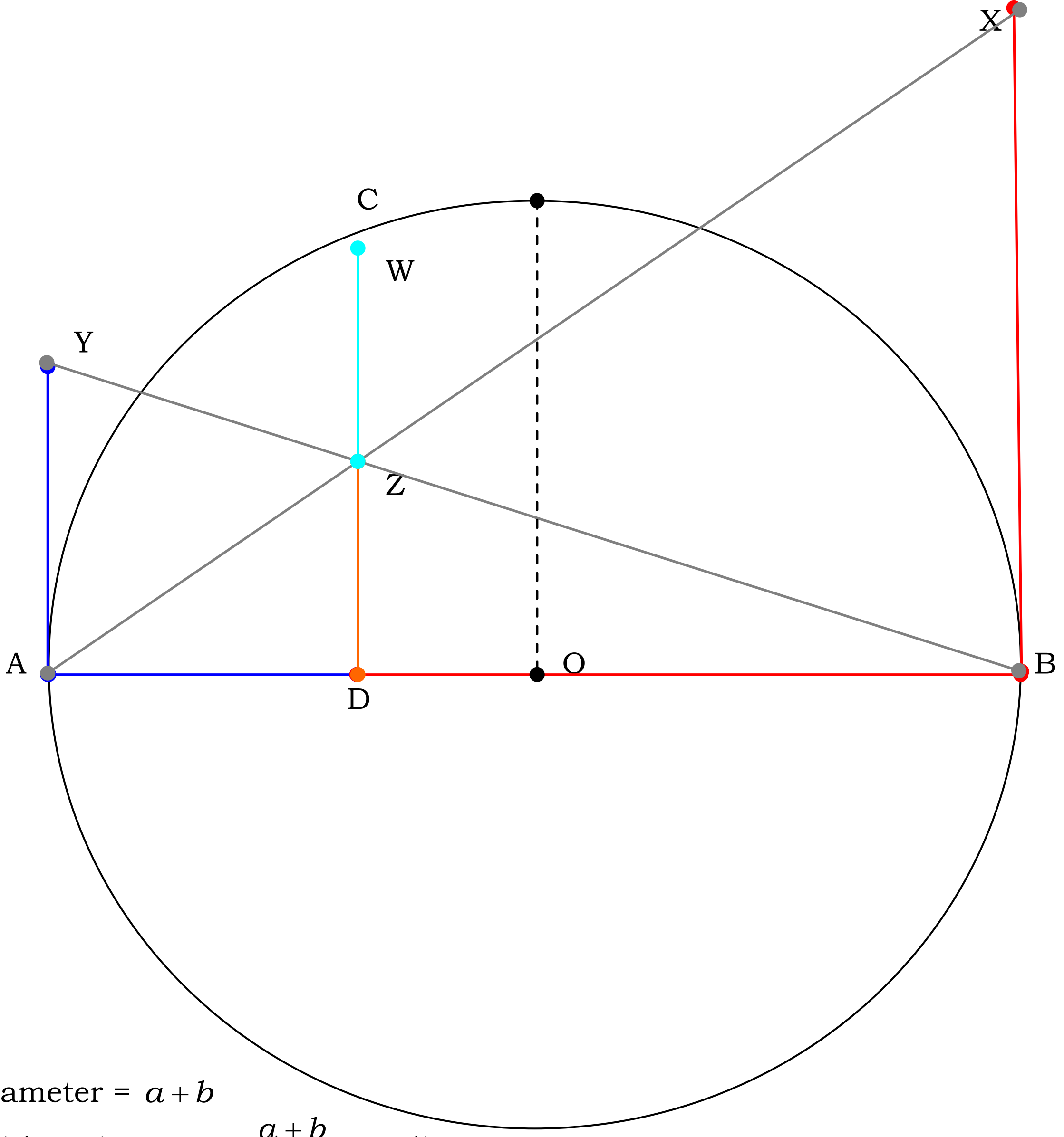
Arithmetic Mean = $\frac{a+b}{2}$ = radius ------

Geometric Mean = $\sqrt{ab}$ = altitude to hypotenuse for ΔABC 🟩

$$\therefore \ \mathbf{AM} \geq \mathbf{GM}$$

Geometric Representation of Harmonic Mean

$\overline{AY}$ is a copy of $\overline{AD}$ and is $\perp$ to $\overline{AD}$. $\overline{XB}$ is a copy of $\overline{BD}$ and is $\perp$ to $\overline{BD}$. $\overline{XA}$ & $\overline{BY}$ are drawn, intersecting at *Z*, and $\overline{ZD}$ is drawn $\perp$ to $\overline{AB}$. By the "two-pole" principle, *ZD* is half the harmonic mean of *EY* and *XF*. $\overline{ZD}$ is extended to twice its length to W , and *DW* is the desired harmonic mean of $\overline{AD}$ & $\overline{DB}$, of a and b .



Diameter = $a+b$

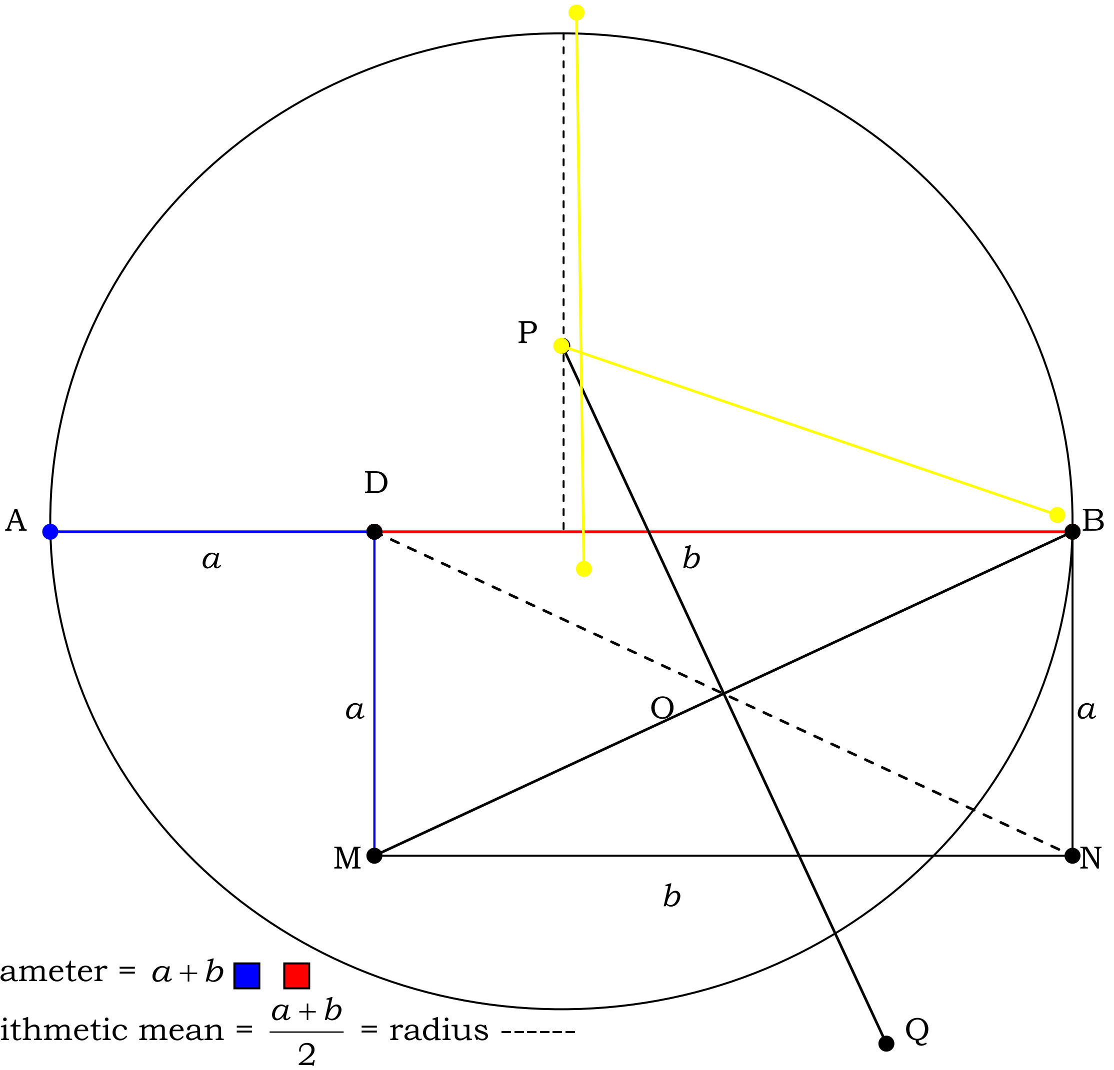
Arithmetic mean = $\dfrac{a+b}{2}$ = radius ------

Harmonic mean = $\dfrac{2ab}{a+b}$ = twice height of intersection (two-pole) 🟧 🟦

Let $\overrightarrow{DC}$ meet ⊙ *O* at *F*. DF is GM $\therefore$ AM $\geq$ GM $\geq$ HM .

Geometric Representation of Root Mean Square

$\square DMNB$, dimensions $a \times b$, is constructed with diagonals, which bisect each other. $\overline{MB}$ is copied and located as $\overline{PQ}$, $\perp$ bisector to $\overline{MB}$. $MB = PQ = \sqrt{a^2+b^2}$. Each side of $\square PMQB$ is $\frac{\sqrt{a^2+b^2}}{\sqrt{2}}$, **RMS** of a and b. Side $\overline{PB}$ is drawn in yellow and copied beside smaller radius = **AM**.



Diameter = $a+b$ 🟦 🟥

Arithmetic mean = $\frac{a+b}{2}$ = radius ------

Geometric mean = $\sqrt{ab}$ = altitude to hypotenuse for ΔABC

Harmonic mean = $\frac{2ab}{a+b}$ = twice height of intersection (two-pole)

Root mean square = $\sqrt{\frac{a^2+b^2}{2}} = \frac{\sqrt{a^2+b^2}}{\sqrt{2}}$ = side of square with diagonal

$= \sqrt{a^2+b^2}$ 🟨 $\therefore$ **RMS** $\geq$ **AM** $\geq$ **GM** $\geq$ **HM**

Reciprocal Function vs. Inverse Function for Trigonometry

Dr. Stan Hartzler Archer City High School

Reciprocal vs. inverse function confusion is easily acquired in trigonometry. In algebra, we think of reciprocal and multiplicative inverse in the same breath, or should.

A. **Basic function** distinction

I. **Inverse functions** for $y = f(x)$ are what you get when

- a. Switching x and y in functions and solving for y
- b. Switching the x and y in ordered pairs like (–5, 20) or in charts
- c. Reflecting a graph across the $y = x$ line (at 45°)
- d. Writing $f^{-1}(x)$

II. **Reciprocal functions** for $y = f(x)$ are what you get when

- a. Writing a fraction with numerator = 1 and denominator = $f(x)$.
- b. On graphs, switching asymptotes with zeroes and switch $0 < y < 1$ with $\infty > y > 1$, and similarly below the x–axis
- c. Writing $(f(x))^{-1}$

B. **Trigonometric function** distinction

I. **Inverses of trig functions** are written like

$\sin^{-1}(x) = \arcsin(x)$. A principal value is $\text{Arcsin}(x)$.

II. **Reciprocal functions** are *renamed.*

$(\sin x)^{-1}$ is called cosecant $= \dfrac{hypotenuse}{opposite}$

$(\cos x)^{-1}$ is called secant $= \dfrac{hypotenuse}{adjacent}$

$(\tan x)^{-1}$ is called cotangent $= \dfrac{adjacent}{opposite}$

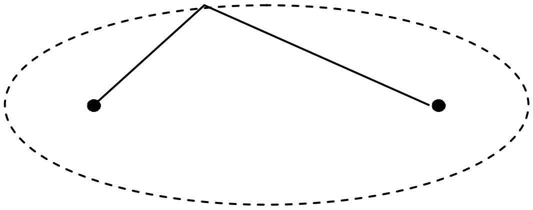
Conics “Constructed” by “Manipulatives”

Dr. Stan Hartzler Archer City High School

A. Circle: Use compass.

B. Ellipse:

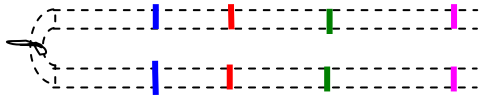
1. Anchor two nails or hooks on a flat surface. There are fixed points. Distance between is $2c$.
2. Tie ends of string to hooks. Length of string between hooks is 2a.
3. Tighten string with tip of pencil, keeping string flat on surface.
4. Move string per directions above as far as possible in either direction, on both sides of the segment joining the fixed points.

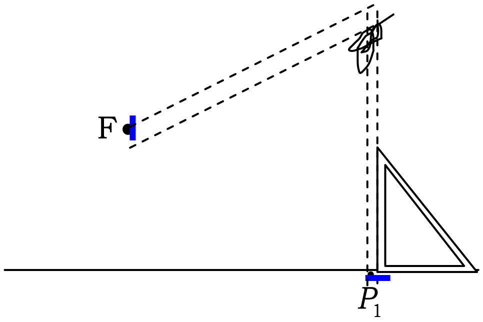


C. Parabola:

Materials: string, marker, and triangle or other drafting device for drawing perpendiculars.

1. Draw a point F (focus) and a line (directrix) on a surface.
2. Prepare a string with a knot in or near the middle, and markings at equal distances from the knots.





3. Place one blue mark on the focus point.
4. Place triangle on line as shown, so that
   - the second blue mark is on the line,
   - the string is perpendicular to the line (per triangle), and
   - the knot is extended fully away from both blue marks.
5. Mark parabola point where knot is.
6. Repeat for other marks.

“Construction” option: 1. Draw FP_1 and perpendicularly bisect it.

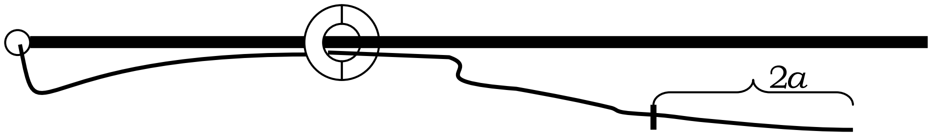
2. Move and adjust compass so that compass point Q_1 is on $\perp$ bisector and distance from to F equals distance from Q_1 to P_1. Mark Q_1.

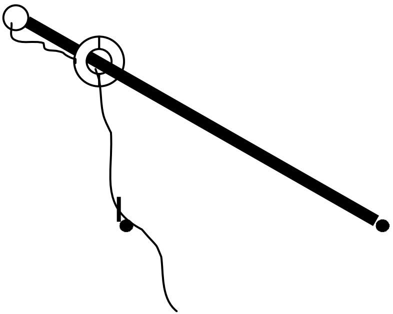
3. Choose another P_2 and find another Q_2

D. Hyperbola

Materials: Stick (18-24 inches long) with anchor (eyelet?) at end, string that is length of stick, washer (or rubber band).

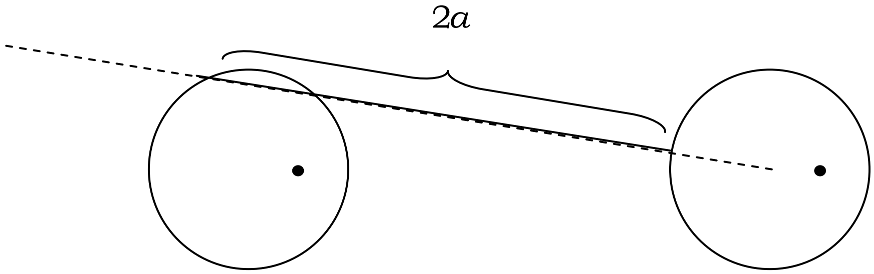
Assemble string, stick, and washer (or rubber band)as shown:





1. Mark string at a point that is $2a$ from the end of the string.
2. Draw two fixed points (focus points) on a flat surface. Distance between is $2c$, where $c > a$.
3. Anchor string mark to one focus point. Drawing has the left point as the string anchor.
4. Hold end of stick at other focus point.
5. Tighten string with washer or by stick rotation, keeping string flat on surface.
6. Mark washer location. This is a point on hyperbola.
7. Move washer and rotate stick to new locations.

"Construction" option: 1. Draw equal circles centered at focus points.
2. Place ruler so that edge is at right focus point and ruler mark at its circle is clear.
3. Measure a $2a$ length from circle to other (left-side) circle, adjusting ruler as needed.
4. When $2a$ length has been found from right circle to left circle, mark point on left circle. This point is a hyperbola point.
5. Repeat for circles of other radii.



To construct auxiliary rectangle,
(1) Construct line joining focus points, and its perpendicular bisector.
(2) Construct right triangle with leg a and hypotenuse c . Other leg = b . (3) Construct two segments of length $2b$ $\perp$ to segment joining focus points, at each vertex and with each vertex the midpoint. Ends of these two segments determine auxiliary rectangle.