Geometric Representation of Means

after Dr. Titu Andrescu

Given two quantities *a* and *b* represented by segments of these respective lengths, the means can be represented and compared geometrically. The means are identified as follows:

- arithmetic mean AM = $\frac{a+b}{2}$
- geometric mean GM = $\sqrt{ab}$
- harmonic mean HM = $\frac{2ab}{a+b}$
- root mean square RMS = $\sqrt{\frac{a^2+b^2}{2}} = \frac{\sqrt{a^2+b^2}}{\sqrt{2}}$

Sketchpad ® steps follow:

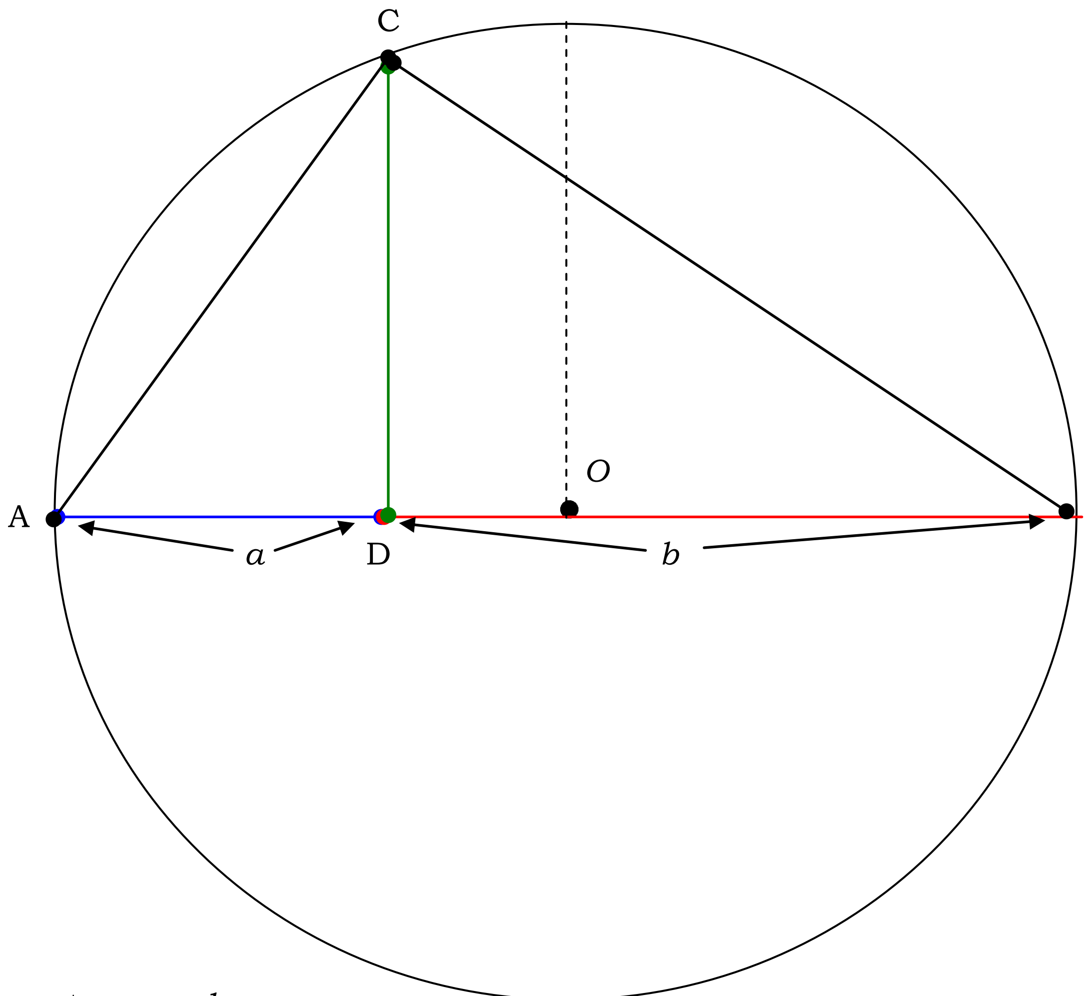
1. Draw diameter $\overline{AB}$.
2. Construct midpoint at C
3. Construct circle, center at C and radius controller at B.
4. Locate D on $\overline{AB}$, not at C.
5. Construct $\perp$ at C.
6. Construct intersection point of circle and $\perp$ at E.
7. Construct $\overline{CE}$ and hide perpendicular line. CE = AM
8. Construct perpendicular at D.
9. Construct intersection point of circle and $\perp$ at H.
10. Construct $\overline{DH}$ and hide perpendicular line. DH = GM, as ΔAHB is a right triangle.
11. Copy $\overline{AD}$ and rotate it $90°$ about center *A*.
12. Copy $\overline{BD}$ and rotate it $270°$ about center *B*.
13. Construct "guy wire" segments $\overline{AQ}$ & $\overline{PB}$ and intersection point F.

Why this is on $\overline{DH}$ is a mystery at present. The length of $\overline{DF}$ is half the HM as in the Two Pole problem.

14. Copy $\overline{DF}$ and rotate the copy $180°$ with center F. $\overline{GD}$ = HM
15. Copy $\overline{DB}$ and rotate the copy $270°$ with center D. New end is L.
16. Construct $\overline{AL}$. $AL = \sqrt{a^2+b^2}$
17. Construct midpoint of $\overline{AL}$. Select the midpoint as a center of rotation.
19. Copy $\overline{AL}$ and rotate it $90°$ though center. New endpoints are P and J. Why J is on circle at end of radius from D is a mystery at present.
20. $DJ = JM = MP = PD = \sqrt{\frac{a^2+b^2}{2}} = \frac{\sqrt{a^2+b^2}}{\sqrt{2}}$ = RMS, longer than AM radius. Conclusion: RMS $\geq$ AM $\geq$ GM $\geq$ HM .

Geometric Representation of Geometric Mean

With ΔABC having $\overline{AB}$ as a diameter of $\odot O$, $\angle ACB$ is a right angle and $\overline{CD}$ is an altitude to hypotenuse $\overline{AB}$. With similar triangles and $\frac{\text{short leg}}{\text{long leg}}$ scheme, $\frac{AD}{DC} = \frac{DC}{CB}$, and CD is **GM** of AD and DB.



Diameter = $a + b$

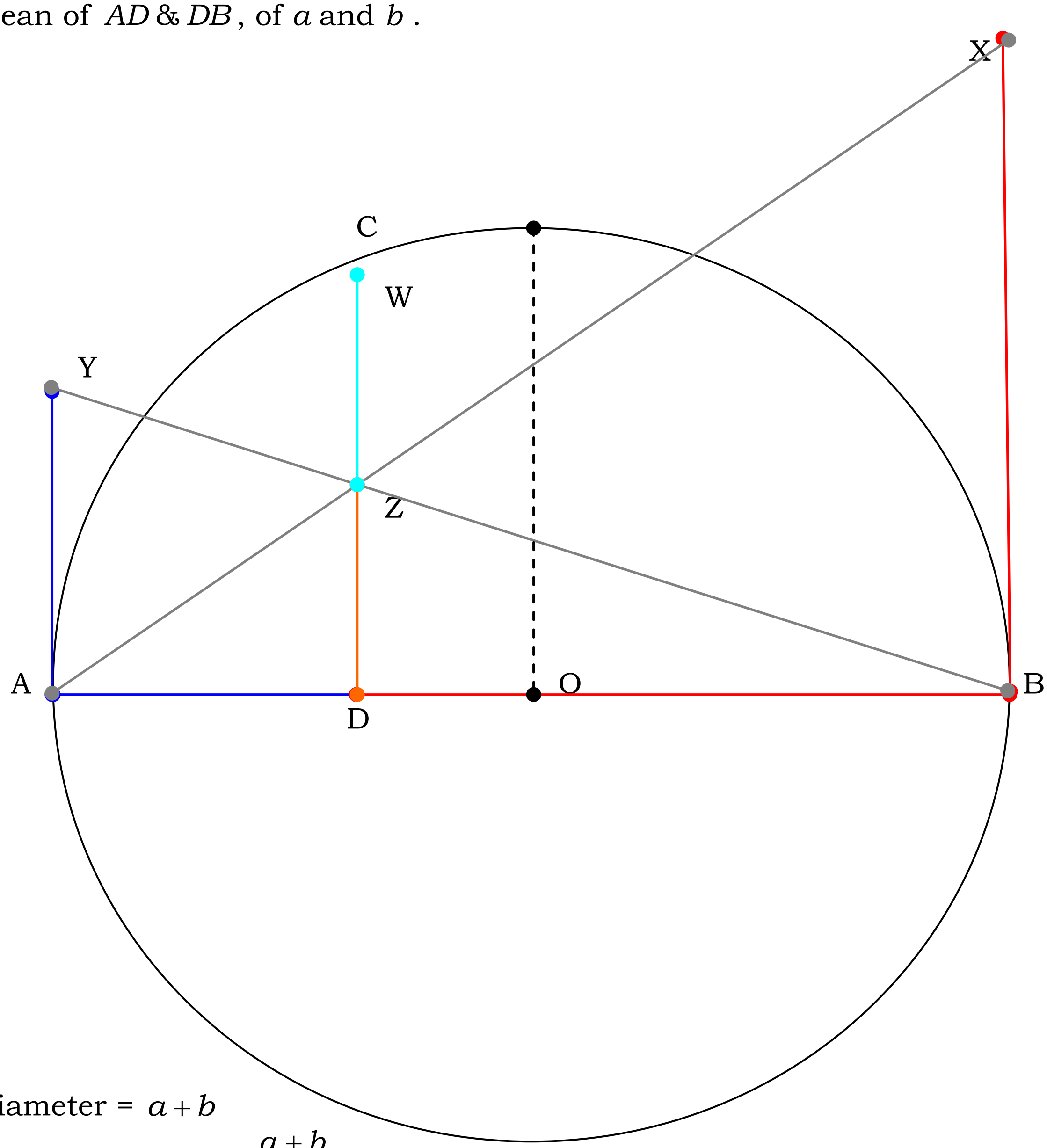
Arithmetic Mean = $\frac{a+b}{2}$ = radius ------

Geometric Mean = $\sqrt{ab}$ = altitude to hypotenuse for ΔABC ■

$\therefore$ **AM** $\geq$ **GM**

Geometric Representation of Harmonic Mean

$\overline{AY}$ is a copy of $\overline{AD}$ and is ⊥ to $\overline{AD}$. $\overline{XB}$ is a copy of $\overline{BD}$ and is ⊥ to $\overline{BD}$. $\overline{XA}$ & $\overline{BY}$ are drawn, intersecting at *Z*, and $\overline{ZD}$ is drawn ⊥ to $\overline{AB}$. By the "two-pole" principle, *ZD* is half the harmonic mean of *EY* and *XF*. $\overline{ZD}$ is extended to twice its length to *W* , and *DW* is the desired harmonic mean of $\overline{AD}$ & $\overline{DB}$, of *a* and *b* .



Diameter = $a+b$

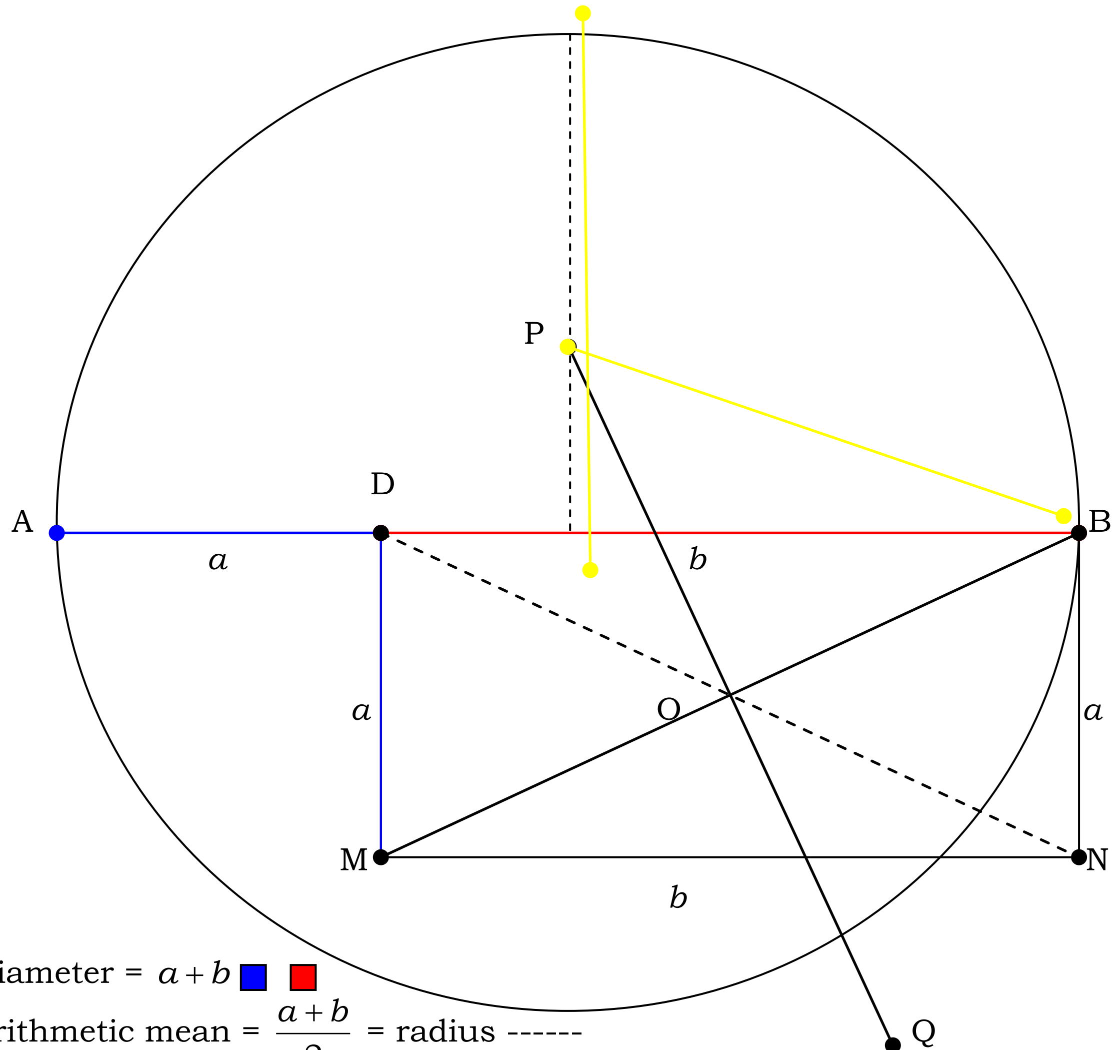
Arithmetic mean = $\dfrac{a+b}{2}$ = radius ------

Harmonic mean = $\dfrac{2ab}{a+b}$ = twice height of intersection (two-pole)

Let $\overrightarrow{DC}$ meet ⊙ *O* at *F*. DF is GM   ∴ AM ≥ GM ≥ HM .

Geometric Representation of Root Mean Square

▭$DMNB$, dimensions $a \times b$, is constructed with diagonals, which bisect each other. $\overline{MB}$ is copied and located as $\overline{PQ}$, $\perp$ bisector to $\overline{MB}$. MB = PQ = $\sqrt{a^2+b^2}$. Each side of ▫$PMQB$ is $\frac{\sqrt{a^2+b^2}}{\sqrt{2}}$, **RMS** of a and b. Side $\overline{PB}$ is drawn in yellow and copied beside smaller radius = **AM**.



Diameter = $a+b$ ■ ■

Arithmetic mean = $\frac{a+b}{2}$ = radius ------

Geometric mean = $\sqrt{ab}$ = altitude to hypotenuse for ΔABC

Harmonic mean = $\frac{2ab}{a+b}$ = twice height of intersection (two-pole)

Root mean square = $\sqrt{\frac{a^2+b^2}{2}} = \frac{\sqrt{a^2+b^2}}{\sqrt{2}}$ = side of square with diagonal $= \sqrt{a^2+b^2}$ ■

$\therefore$ **RMS** $\geq$ **AM** $\geq$ **GM** $\geq$ **HM**