Connecting Whole Number Operations

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<u>Addition -- Subtraction -- Multiplication</u>

For addition, initial elements are named the <u>same</u>: *addends*. Order doesn't matter in any respect.
For subtraction, order of the elements matters in every respect. The initial elements of subtraction have <u>different</u> names: *minuend* and *subtrahend*.

$$\begin{array}{rl} 21020 & \text{minuend} \\ \underline{-\ \ 4273} & \text{subtrahend} \\ 16747 & \text{difference} \end{array}$$

Addition and subtraction elements are ***always the same*** kinds. We add apples to apples, or oranges to oranges. Consequently,

- for whole number and decimal fraction addition or subtraction, elements are lined up to add ones to ones, tenths to tenths, etc. *Position matters*.
- common (same) denominators are used for fraction addition/subtraction.

By contrast, the elements of multiplication are ***never the same***. Recalling Talton's TAPS outline, multiplication involves combining groups of equal size.

For $\begin{array}{r} 54 \\ \underline{\times\ \ 3} \end{array}$

54 multiplicand -- tells number <u>in each group</u> being combined.
× 3 multiplier -- tells number <u>of groups</u> being combined.

So 3×54 means $54 + 54 + 54$. Written in column form: $\begin{array}{r} 54 \\ \underline{\times\ \ 3} \end{array} = \begin{array}{r} 54 \\ 54 \\ \underline{+\ 54} \end{array}$

This meaning justifies why we distribute -- yes, distribute, the preview to distributivity in algebra, where $2(x + 5) = 2x + 10$.

The three is multiplied times <u>both</u> the 4 and the 50, *regardless of position*.

When these exercises are gearshifted with 54 + 3, the student may be tempted to add three to the 4 and to the 50 as well. If the student asks why not (good!), this response can be made: "When we have 54 and 3 more, we have 57 in all.

$$\begin{array}{r} 54 \\ \underline{+\ \ 3} \end{array}$$

" But when we *multiply* 3×54, we have three 4's and three 50's, as in the column addition illustration. Do you see why?"

This gearshifting needs to continue for many days after initial instruction.

Terminology for horizontal form helps establish why "of" means ***multiply*** in word-problem situations. "Three groups of 54…"

$3 \times 54 =$ multiplicand (number in each group) → $\begin{array}{r} 54 \\ \times\ \ 3 \\ \text{---------} \\ 162 \end{array}$

multiplier (number of groups)

product (total in all groups)

There are two important short-cuts in whole-number multiplication:

a) 5400 × 381 is re-written

$$\begin{array}{r} 3\,8\,1 \\ \times\ \ 5\,4\,0\,0 \\ \hline 1\,5\,2\,4 \\ 1\,9\,0\,5\,\_ \\ \hline 2\,0\,5\,7\,4\,0\,0 \end{array}$$

b) 304 × 111 is written

$$\begin{array}{r} 1\,1\,1 \\ \times\,3\,0\,4 \\ \hline 4\,4\,4 \\ 3\,3\,3\,\_\,\_ \\ \hline 3\,3\,7\,4\,4 \end{array}$$

These shortcuts save time, emphasize that addition with zero is silly, and minimize errors in lining up digits. *Irrelevance* of position is also shown.

Is multiplication <u>really commutative</u>? (4 × 3 = 3 × 4?) **YES and NO!**

- Four classes with 50 students in each involves 200 students.
- Fifty classes with four students in each also involves 200 students.

But what a difference to the school!

Multiplication <u>is</u> commutative with respect to <u>answers</u>. It is NOT commutative with respect to meaning and situations. The first example above would be written 4 × 50, and the second, 50 × 4. This distinction is made in elementary textbooks.

By contrast, a*ddition <u>is</u> commutative in every respect.*

Coefficients in algebra can be generalized from multiplication meaning, horizontal form:

$$\text{If } 3 + 3 + 3 + 3 = 4 \times 3, \text{ then } a + a + a + a = 4 \times a = 4\cdot a$$

Once again, great arithmetic learning helps the algebra learning.

<u>**Division**</u>

<u>Language & symbols</u>: 6 ÷ 3 means 6 : 3 or $\frac{6}{3}$ or $3\overline{)6}$ -- all of which also mean "6 divided by 3" or "3 goes into 6" or "3 parts of 6"

<u>Remainders</u> are expressed in three ways. All remain important, even as the most basic way is joined by the two more sophisticated ways.

<u>Why division by zero is impossible</u>:

- 8 ÷ 4 = 2 because 2 × 4 = 8.
- If 8 ÷ 0 = x, then x × 0 = 8 for some x. Impossible!

<u>Processes</u>

D
M
S
BD

$$\begin{array}{r} 1760 \\ 3\overline{)5280} \\ \underline{3} \\ 22 \\ \underline{21} \\ 18 \\ \underline{18} \\ 00 \end{array}$$

Only the across-the-top method (shown here) should be taught, as it is more useful and direct, and extends into algebra and beyond. Admittedly, the down-the-side "Greenwood" method is more useful for basic division <u>understanding</u> in lower grades.

Many students know only the down-the-side method, and usually only vaguely. This writer always teaches the across-the-top process to such students. Many students have the two systems confused. They and those knowing no method at all are usually helped by the vertical list of initials for steps, arrow included, also shown here.

These initials indicate the four steps, Divide, Multiply, Subtract, and Bring-Down; the arrow indicates that the steps are repeated as needed. One teacher in this course also recommended a "test" step between subtract and bring-down, to insure that the difference was less than the divisor. AMEN!

A second memory trick used at one lab school consists of the question, "Does McDonald's Sell Big Delicious Milk Shakes…?", the first letters of the words cycling students through the loop of steps and back into the beginning again.

This process is best in decimal division, improper fraction conversion, percent, and algebraic division of polynomials, a skill needed in calculus and beyond. It also promotes estimation and perseverance. This topic often provides a student's first learning to control self, and to work.

Experience in teaching long division reveals that after the sequence of steps is learned, stumbling blocks appear: carrying with multiplying, borrowing when subtracting, and zeroes in the quotient. Included with this article are problem sets designed to help remediate such complications. These exercises also constitute a model for dealing with stumbling points in other topics.

Also included is a self-contained diagnosis-remediation system. Students do only the first exercise in each row. If an exercise is missed, the teacher gives corrective feedback and assigns the rest of the row to practice. If the first in a row is done correctly, the rest of the row can be skipped. The teacher is invited to try a few problems to see what error traps are involved.

Also included: a sequence of pages designed to shoe-horn the student past sticking points, described here:

- Part 1 (the first nine problems) is devoid of any carrying or borrowing.
- Part 2 involves carrying but no borrowing, and vice versa for part 3.
- Part 4 involves both. Part 5 is devoid of carrying and borrowing, but there is a zero in each quotient.
- Parts 6-8 reintroduce carrying and borrowing, along with quotient zeroes.

This sorting is contrary to the gearshifting idea presented elsewhere in this teacher's writings. Certainly gearshifting of the varied types must follow these exercise sets. The sorting shown here was helpful, however, for students who first needed remediation for multiplication, and this strategy helped them develop confidence in playing defense with the nasty issues of division.

Students notably weak in confidence with complexity, especially special education students, often prosper with these topical breakdowns.

Terminology: $$\begin{array}{r} \text{quotient} \\ \text{divisor}\,\overline{)\,\text{dividend}} \end{array}$$

The dividend is the grand total.

The divisor may be the number in each group, which then means the quotient is the number of groups. But, on the other hand, the divisor may be the number of groups, by which the quotient is the number in each group.

A manipulative illustration of the steps of the standard algorithm is given here.

WHOLE-NUMBER DIVISION: A MANIPULATIVE ILLUSTRATION

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Four children divide 381 cents to illustrate long division.

Part I: Four students sit around a round table with 381 pennies in the middle. The rest of the class stands around (behind) and observes. The four students are instructed to take a penny each time the teacher says, “Take.”

1. How many times will the teacher say take?
2. How many pennies will each of the four students get?
3. Will all of the pennies be taken?

Part II: Using dollars to represent hundreds, dimes to represent tens, and pennies to represent ones, the teacher starts again, saying, "When I say take, each of you four students please take a dollar."

What will happen?

What must be done so that the students may proceed to divide the money? This answer will have several pieces.

$$\begin{array}{r} 95 \\ 4\,\overline{)\,381} \\ \underline{36} \\ 21 \\ \underline{20} \\ 1 \end{array}$$

Looking at the pencil-paper process NOW:

4. The one at the bottom tells what ?
5. The five in the quotient tells what?
6. The 21 tells what?
7. The 20 under the 21 tells what?
8. The two that results from the first subtraction tells what?

For the above, tell a) number of groups, and b) number in each group.

<u>Learning Question 1</u>: Should the algorithm be taught first, then the manipulation, or should it be the other way, or should we go for both at once? Or skip it?
<u>Learning Question 2</u>: This writer says that some manipulatives are great for nearly everyone (area squares), other ideas are impossible to illustrate with manipulatives, and some ideas may or may not be helped by manipulatives, depending on characteristics of students and topic. One rule of the thumb, however, holds for all topics: don't buy manipulatives from a catalog until you've thought about using a manipulative you already own -- rubber bands, or toothpicks, or whatever. How does the illustration of this page fit into that admonition?

A mathematics education professor in Iowa, Patsy Fagan, recently endorsed this illustration as a way to introduce the algorithm. In tune with Evans and Carnine's finding for later illustration as *more efficient* in subtraction, it may be wiser to consider the pencil-paper first. Other teachers have suggested that such illustrations are too confusing to be of any value for <u>either</u> initial introduction or follow-up illustration.

Language and Symbols (AGAIN) Several symbolic or verbal ways of expressing division are often confused. Here is one consistent inventory:

$6 \div 3$ $6 : 3$ 3 goes into 6   6 divided by 3 $\frac{6}{3}$ $3\overline{)6}$ 3 parts of 6

Three ways to express remainders are illustrated here:

$$6 \div 4 = 1 \text{ r } 2 \qquad 6 \div 4 = 1\frac{1}{2} \qquad 6 \div 4 = 1.5$$

All three ways have their permanent place, never to be outmoded. The first method is needed in problems such as this:

> Forty-one cows are divided into nine groups of equal size. How many cows are in each group?

Division Practice Tricky Remainders and Quotient Zeroes (6c)

Directions: Do column A only. Ask your teacher to check your work. If your answer is correct, you may skip the rest of that row. If you make a mistake, your teacher will show you the correct process, and then you will practice the rest of that row.

	A	**B**	**C**	**D**	**E**
1)	$2\overline{)6001}$	$7\overline{)7006}$	$3\overline{)9002}$	$4\overline{)4002}$	$6\overline{)6003}$
2)	$5\overline{)4537}$	$7\overline{)5648}$	$9\overline{)6329}$	$8\overline{)7219}$	$4\overline{)3531}$
3)	$2\overline{)6001}$	$7\overline{)7006}$	$3\overline{)9002}$	$4\overline{)4002}$	$6\overline{)6003}$
4)	$7\overline{)43,406}$	$6\overline{)39,005}$	$8\overline{)59,206}$	$3\overline{)29,402}$	$5\overline{)47,501}$
5)	$4\overline{)80,323}$	$9\overline{)90,546}$	$3\overline{)60,242}$	$5\overline{)50,404}$	$7\overline{)70,355}$

Division Practice Progression from Basic to Complexity Parts 1 and 2

$16\overline{)1776}$ $14\overline{)2954}$ $23\overline{)4853}$

$31\overline{)9951}$ $41\overline{)8651}$ $26\overline{)2886}$

$22\overline{)6864}$ $11\overline{)6798}$ $12\overline{)4968}$

$16\overline{)3696}$ $15\overline{)3480}$ $18\overline{)7776}$

$13\overline{)6799}$ $19\overline{)9899}$ $24\overline{)9888}$

$17\overline{)8891}$ $25\overline{)5775}$ $14\overline{)9996}$

Division Practice Progression from Basic to Complexity Parts 3 and 4

$12\overline{)5076}$	$13\overline{)3003}$	$21\overline{)9114}$
$42\overline{)8904}$	$32\overline{)7424}$	$23\overline{)7659}$
$24\overline{)2928}$	$31\overline{)7223}$	$33\overline{)7326}$

$68\overline{)8228}$	$45\overline{)9720}$	$38\overline{)5016}$
$44\overline{)7216}$	$56\overline{)7560}$	$29\overline{)7627}$
$17\overline{)8313}$	$32\overline{)5024}$	$86\overline{)9804}$

Division Practice Progression from Basic to Complexity Parts 5 and 6

$30\overline{)9060}$	$20\overline{)2020}$	$10\overline{)4030}$
$44\overline{)8844}$	$21\overline{)6384}$	$12\overline{)4836}$
$27\overline{)2727}$	$14\overline{)2828}$	$33\overline{)9966}$

$19\overline{)3857}$	$28\overline{)5684}$	$17\overline{)3485}$
$23\overline{)9246}$	$29\overline{)8729}$	$18\overline{)1890}$
$14\overline{)8498}$	$16\overline{)9680}$	$13\overline{)9178}$

Division Practice Progression from Basic to Complexity Parts 7 and 8

$16\overline{)1760}$ $32\overline{)9920}$ $14\overline{)1680}$

$14\overline{)2940}$ $41\overline{)8610}$ $72\overline{)7920}$

$26\overline{)2860}$ $23\overline{)2990}$ $30\overline{)9600}$

$26\overline{)5980}$ $24\overline{)7680}$ $16\overline{)8960}$

$23\overline{)9890}$ $17\overline{)8840}$ $18\overline{)5760}$

$27\overline{)8910}$ $19\overline{)9880}$ $28\overline{)8960}$

Division Practice Progression from Basic to Complexity Parts 9 and 10

$44\overline{)9240}$ $23\overline{)5290}$ $31\overline{)4030}$

$22\overline{)7040}$ $33\overline{)7590}$ $32\overline{)7360}$

$24\overline{)5280}$ $73\overline{)8030}$ $42\overline{)5040}$

$26\overline{)7020}$ $29\overline{)7540}$ $19\overline{)4180}$

$34\overline{)6120}$ $34\overline{)9180}$ $16\overline{)5120}$

$28\overline{)3360}$ $14\overline{)2100}$ $38\overline{)4180}$