Connecting Whole Number Operations

Dr. Stan Hartzler Archer City High School

Addition -- Subtraction -- Multiplication

For addition, initial elements are named the <u>same</u>: *addends*. Order doesn't matter in any respect.
For subtraction, order of the elements matters in every respect. The initial elements of subtraction have <u>different</u> names: *minuend* and *subtrahend*.

$$\begin{array}{rl} 21020 & \text{minuend} \\ -\ \ 4273 & \text{subtrahend} \\ \hline 16747 & \text{difference} \end{array}$$

Addition and subtraction elements are ***always the same*** kinds. We add apples to apples, or oranges to oranges. Consequently,

- for whole number and decimal fraction addition or subtraction, elements are lined up to add ones to ones, tenths to tenths, etc. *Position matters*.
- common (same) denominators are used for fraction addition/subtraction.

By contrast, the elements of multiplication are ***never the same***. Recalling Talton's TAPS outline, multiplication involves combining groups of equal size.

For

$$\begin{array}{rl} 54 & \text{multiplicand -- tells number \underline{in each group} being combined.} \\ \underline{\times\ \ 3} & \text{multiplier -- tells number \underline{of groups} being combined.} \end{array}$$

So 3 × 54 means 54 + 54 + 54 . Written in column form:

$$\begin{array}{r} 54 \\ \underline{\times\ \ 3} \end{array} = \begin{array}{r} 54 \\ 54 \\ \underline{+\ 54} \end{array}$$

This meaning justifies why we distribute -- yes, distribute, the preview to distributivity in algebra, where $2(x + 5) = 2x + 10$.

The three is multiplied times <u>both</u> the 4 and the 50, *regardless of position*.

When these exercises are gearshifted with 54 + 3, the student may be tempted to add three to the 4 and to the 50 as well. If the student asks why not (good!), this response can be made: "When we have 54 and 3 more, we have 57 in all.

$$\begin{array}{r} 54 \\ \underline{+\ \ 3} \end{array}$$

" But when we *multiply* 3 × 54, we have three 4's and three 50's, as in the column addition illustration. Do you see why?"

This gearshifting needs to continue for many days after initial instruction.

Terminology for horizontal form helps establish why "of" means ***multiply*** in word-problem situations. "Three groups of 54..."

3 × 54 = multiplicand → 54
(number in each group)

× 3

162

multiplier
(number of groups)

product (total in all groups)

There are two important short-cuts in whole-number multiplication:

a) 5400 × 381 is re-written

```
      3 8 1
    ×   5 4 0 0
  ------------
     1 5 2 4
   1 9 0 5 _
  -------------
  2 0 5 7 4 0 0
```

b) 304 × 111 is written

```
     1 1 1
   × 3 0 4
    --------
     4 4 4
   3 3 3 _ _
    -----------
   3 3 7 4 4
```

These shortcuts save time, emphasize that addition with zero is silly, and minimize errors in lining up digits. *Irrelevance* of position is also shown.

Is multiplication <u>really commutative</u>? ($4 \times 3 = 3 \times 4$?) **YES and NO!**

- Four classes with 50 students in each involves 200 students.
- Fifty classes with four students in each also involves 200 students.

But what a difference to the school!

Multiplication <u>is</u> commutative with respect to <u>answers</u>. It is NOT commutative with respect to meaning and situations. The first example above would be written 4×50, and the second, 50×4. This distinction is made in elementary textbooks.

By contrast, a*ddition <u>is</u> commutative in every respect.*

Coefficients in algebra can be generalized from multiplication meaning, horizontal form:

$$\text{If } 3 + 3 + 3 + 3 = 4 \times 3, \text{ then } a + a + a + a = 4 \times a = 4 \cdot a$$

Once again, great arithmetic learning helps the algebra learning.

<u>Division</u>

<u>Language & symbols</u>: $6 \div 3$ means $6 : 3$ or $\frac{6}{3}$ or $3\overline{)6}$ -- all of which also mean "6 divided by 3" or "3 goes into 6" or "3 parts of 6"

<u>Remainders</u> are expressed in three ways. All remain important, even as the most basic way is joined by the two more sophisticated ways.

<u>Why division by zero is impossible</u>:

- $8 \div 4 = 2$ because $2 \times 4 = 8$.
- If $8 \div 0 = x$, then $x \times 0 = 8$ for some x. Impossible!

Processes

D
M
S
BD

$$\begin{array}{r} 1760 \\ 3\overline{)5280} \\ \underline{3} \\ 22 \\ \underline{21} \\ 18 \\ \underline{18} \\ 00 \end{array}$$

Only the across-the-top method (shown here) should be taught, as it is more useful and direct, and extends into algebra and beyond. Admittedly, the down-the-side "Greenwood" method is more useful for basic division understanding in lower grades.

Many students know only the down-the-side method, and usually only vaguely. This writer always teaches the across-the-top process to such students. Many students have the two systems confused. They and those knowing no method at all are usually helped by the vertical list of initials for steps, arrow included, also shown here.

These initials indicate the four steps, Divide, Multiply, Subtract, and Bring-Down; the arrow indicates that the steps are repeated as needed. One teacher in this course also recommended a "test" step between subtract and bring-down, to insure that the difference was less than the divisor. AMEN!

A second memory trick used at one lab school consists of the question, "Does McDonald's Sell Big Delicious Milk Shakes...?", the first letters of the words cycling students through the loop of steps and back into the beginning again.

This process is best in decimal division, improper fraction conversion, percent, and algebraic division of polynomials, a skill needed in calculus and beyond. It also promotes estimation and perseverance. This topic often provides a student's first learning to control self, and to work.

Experience in teaching long division reveals that after the sequence of steps is learned, stumbling blocks appear: carrying with multiplying, borrowing when subtracting, and zeroes in the quotient. Included with this article are problem sets designed to help remediate such complications. These exercises also constitute a model for dealing with stumbling points in other topics.

Also included is a self-contained diagnosis-remediation system. Students do only the first exercise in each row. If an exercise is missed, the teacher gives corrective feedback and assigns the rest of the row to practice. If the first in a row is done correctly, the rest of the row can be skipped. The teacher is invited to try a few problems to see what error traps are involved.

Also included: a sequence of pages designed to shoe-horn the student past sticking points, described here:

- Part 1 (the first nine problems) is devoid of any carrying or borrowing.
- Part 2 involves carrying but no borrowing, and vice versa for part 3.
- Part 4 involves both. Part 5 is devoid of carrying and borrowing, but there is a zero in each quotient.
- Parts 6-8 reintroduce carrying and borrowing, along with quotient zeroes.

This sorting is contrary to the gearshifting idea presented elsewhere in this teacher's writings. Certainly gearshifting of the varied types must follow these exercise sets. The sorting shown here was helpful, however, for students who first needed remediation for multiplication, and this strategy helped them develop confidence in playing defense with the nasty issues of division.

Students notably weak in confidence with complexity, especially special education students, often prosper with these topical breakdowns.

Terminology: $\begin{array}{r} \text{quotient} \\ \text{divisor}\overline{\smash{)}\text{dividend}} \end{array}$

The dividend is the grand total.

The divisor may be the number in each group, which then means the quotient is the number of groups. But, on the other hand, the divisor may be the number of groups, by which the quotient is the number in each group.

A manipulative illustration of the steps of the standard algorithm is given here.

WHOLE-NUMBER DIVISION: A MANIPULATIVE ILLUSTRATION

Dr. Stan Hartzler Archer City High School

Four children divide 381 cents to illustrate long division.

Part I: Four students sit around a round table with 381 pennies in the middle. The rest of the class stands around (behind) and observes. The four students are instructed to take a penny each time the teacher says, “Take.”

1. How many times will the teacher say take?
2. How many pennies will each of the four students get?
3. Will all of the pennies be taken?

Part II: Using dollars to represent hundreds, dimes to represent tens, and pennies to represent ones, the teacher starts again, saying, "When I say take, each of you four students please take a dollar."

What will happen?

What must be done so that the students may proceed to divide the money? This answer will have several pieces.

$$\begin{array}{r} 95 \\ 4\overline{\smash{)}381} \\ \underline{36} \\ 21 \\ \underline{20} \\ 1 \end{array}$$

Looking at the pencil-paper process NOW:

4. The one at the bottom tells what ?
5. The five in the quotient tells what?
6. The 21 tells what?
7. The 20 under the 21 tells what?
8. The two that results from the first subtraction tells what?

For the above, tell a) number of groups, and b) number in each group.

<u>Learning Question 1</u>: Should the algorithm be taught first, then the manipulation, or should it be the other way, or should we go for both at once? Or skip it?
<u>Learning Question 2</u>: This writer says that some manipulatives are great for nearly everyone (area squares), other ideas are impossible to illustrate with manipulatives, and some ideas may or may not be helped by manipulatives, depending on characteristics of students and topic. One rule of the thumb, however, holds for all topics: don't buy manipulatives from a catalog until you've thought about using a manipulative you already own -- rubber bands, or toothpicks, or whatever. How does the illustration of this page fit into that admonition?

A mathematics education professor in Iowa, Patsy Fagan, recently endorsed this illustration as a way to introduce the algorithm. In tune with Evans and Carnine's finding for later illustration as *more efficient* in subtraction, it may be wiser to consider the pencil-paper first. Other teachers have suggested that such illustrations are too confusing to be of any value for <u>either</u> initial introduction or follow-up illustration.

Language and Symbols (AGAIN) Several symbolic or verbal ways of expressing division are often confused. Here is one consistent inventory:

6 ÷ 3 6 : 3 3 goes into 6 6 divided by 3 $\frac{6}{3}$ $3\overline{)6}$ 3 parts of 6

Three ways to express remainders are illustrated here:

$6 \div 4 = 1 \text{ r } 2$ $6 \div 4 = 1\frac{1}{2}$ $6 \div 4 = 1.5$

All three ways have their permanent place, never to be outmoded. The first method is needed in problems such as this:

> Forty-one cows are divided into nine groups of equal size. How many cows are in each group?

Division Practice Tricky Remainders and Quotient Zeroes (6c)

Directions: Do column A only. Ask your teacher to check your work. If your answer is correct, you may skip the rest of that row. If you make a mistake, your teacher will show you the correct process, and then you will practice the rest of that row.

	A	**B**	**C**	**D**	**E**
1)	$2\overline{)6001}$	$7\overline{)7006}$	$3\overline{)9002}$	$4\overline{)4002}$	$6\overline{)6003}$
2)	$5\overline{)4537}$	$7\overline{)5648}$	$9\overline{)6329}$	$8\overline{)7219}$	$4\overline{)3531}$
3)	$2\overline{)6001}$	$7\overline{)7006}$	$3\overline{)9002}$	$4\overline{)4002}$	$6\overline{)6003}$
4)	$7\overline{)43,406}$	$6\overline{)39,005}$	$8\overline{)59,206}$	$3\overline{)29,402}$	$5\overline{)47,501}$
5)	$4\overline{)80,323}$	$9\overline{)90,546}$	$3\overline{)60,242}$	$5\overline{)50,404}$	$7\overline{)70,355}$

Division Practice Progression from Basic to Complexity Parts 1 and 2

$16\overline{)1776}$ $14\overline{)2954}$ $23\overline{)4853}$

$31\overline{)9951}$ $41\overline{)8651}$ $26\overline{)2886}$

$22\overline{)6864}$ $11\overline{)6798}$ $12\overline{)4968}$

$16\overline{)3696}$ $15\overline{)3480}$ $18\overline{)7776}$

$13\overline{)6799}$ $19\overline{)9899}$ $24\overline{)9888}$

$17\overline{)8891}$ $25\overline{)5775}$ $14\overline{)9996}$

Division Practice Progression from Basic to Complexity Parts 3 and 4

$12\overline{)5076}$	$13\overline{)3003}$	$21\overline{)9114}$
$42\overline{)8904}$	$32\overline{)7424}$	$23\overline{)7659}$
$24\overline{)2928}$	$31\overline{)7223}$	$33\overline{)7326}$

$68\overline{)8228}$	$45\overline{)9720}$	$38\overline{)5016}$
$44\overline{)7216}$	$56\overline{)7560}$	$29\overline{)7627}$
$17\overline{)8313}$	$32\overline{)5024}$	$86\overline{)9804}$

Division Practice   Progression from Basic to Complexity   Parts 5 and 6

$30\overline{)9060}$	$20\overline{)2020}$	$10\overline{)4030}$
$44\overline{)8844}$	$21\overline{)6384}$	$12\overline{)4836}$
$27\overline{)2727}$	$14\overline{)2828}$	$33\overline{)9966}$

$19\overline{)3857}$	$28\overline{)5684}$	$17\overline{)3485}$
$23\overline{)9246}$	$29\overline{)8729}$	$18\overline{)1890}$
$14\overline{)8498}$	$16\overline{)9680}$	$13\overline{)9178}$

Division Practice Progression from Basic to Complexity Parts 7 and 8

$16\overline{)1760}$ $32\overline{)9920}$ $14\overline{)1680}$

$14\overline{)2940}$ $41\overline{)8610}$ $72\overline{)7920}$

$26\overline{)2860}$ $23\overline{)2990}$ $30\overline{)9600}$

$26\overline{)5980}$ $24\overline{)7680}$ $16\overline{)8960}$

$23\overline{)9890}$ $17\overline{)8840}$ $18\overline{)5760}$

$27\overline{)8910}$ $19\overline{)9880}$ $28\overline{)8960}$

Division Practice Progression from Basic to Complexity Parts 9 and 10

$44\overline{)9240}$ $23\overline{)5290}$ $31\overline{)4030}$

$22\overline{)7040}$ $33\overline{)7590}$ $32\overline{)7360}$

$24\overline{)5280}$ $73\overline{)8030}$ $42\overline{)5040}$

$26\overline{)7020}$ $29\overline{)7540}$ $19\overline{)4180}$

$34\overline{)6120}$ $34\overline{)9180}$ $16\overline{)5120}$

$28\overline{)3360}$ $14\overline{)2100}$ $38\overline{)4180}$

Greatest Common Factor, Lowest Common Multiple

Middle-Grades (+?)

To find Greatest Common Factor and Lowest Common Multiple of 80 and 150:

Factor Lists			
150		**80**	
1	150	1	80
2	75	2	40
3	50	4	20
5	30	5	16
6	25	8	10
10	15		
Greatest **C**ommon **F**actor: 10			

Prime Factorization	
150	80
15 10	8 10
3 5 2 5	2 2 2 2 5
$2 \bullet 3 \bullet 5^2$	$2^4 \bullet 5$
Greatest **C**ommon	**F**actor: $2 \bullet 5$

III. Introducing the "divides" bar.

The statement " 6 | 18 " means "6 divides into 18 without remainder."

<table>
<tr>
<td>To find GCF, a smaller number, write the given numbers on the right of the "divides" bars. The number in the blank must be the biggest collection of factors that will divide into both prime factorizations.</td>
<td>_____ | $2 \bullet 3 \bullet 5^2$ |
_____ | $2^4 \bullet 5$ | _____</td>
<td>To find LCM, a larger number, write the given numbers on the left of the "divides" bar. The number in the blank must be the smallest collection of factors that can be divided by both prime factorizations.</td>
</tr>
</table>

Algebra example: Find GCF (think smaller) and LCM (think larger) for these expressions:

$8a^3xz^2 \qquad 12a^2bx^2z^2$

_____ | $8a^3xz^2$ |

_____ | $12a^2bx^2z^2$ | _____

Division Practice   Progression from Basic to Complexity   Parts 1 and 2

$16\overline{)1776}$ $14\overline{)2954}$ $23\overline{)4853}$

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$30\overline{)9060}$	$20\overline{)2020}$	$10\overline{)4030}$
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$19\overline{)3857}$	$28\overline{)5684}$	$17\overline{)3485}$
$23\overline{)9246}$	$29\overline{)8729}$	$18\overline{)1890}$
$14\overline{)8498}$	$16\overline{)9680}$	$13\overline{)9178}$

Division Practice   Progression from Basic to Complexity   Parts 7 and 8

$16\overline{)1760}$ $32\overline{)9920}$ $14\overline{)1680}$

$14\overline{)2940}$ $41\overline{)8610}$ $72\overline{)7920}$

$26\overline{)2860}$ $23\overline{)2990}$ $30\overline{)9600}$

$26\overline{)5980}$ $24\overline{)7680}$ $16\overline{)8960}$

$23\overline{)9890}$ $17\overline{)8840}$ $18\overline{)5760}$

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Equivalent Proportions

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Equivalent proportions demand attention for a variety of reasons. One special reason is truer in a time of emphasis on cooperative learning.

> Example problem: Given the ratio of three apples to five pears, how many apple are there if there are 120 pieces of fruit?

While setting up this example, one student might write apples above pears, and another might order them the other way. One may choose to organize by writing the "actual" column first and the "ratio" column later. Yet another might decide to make the "ratio" and "actual" columns be rows instead -- all of which is good "bookkeeping" and will produce the same correct result.

BUT! The initial proportions may not look the same, and unless they are familiarized with equivalent proportion ideas, students working in cooperative groups may be inclined to question themselves or their classmates where no question is necessary. In this light, familiarity with equivalent proportions is urged.

One approach might be as follows. ***If the proportion shown here as GIVEN is true, then the other four statements are also true.***

GIVEN	FLIP	DIAL	SWITCH	CROSS ×
$\frac{a}{b}=\frac{c}{d}$	$\frac{b}{a}=\frac{d}{c}$	$\frac{c}{a}=\frac{d}{b}$	$\frac{d}{b}=\frac{c}{a}$	$ad=bc$
$\frac{3}{4}=\frac{6}{8}$	$\frac{4}{3}=\frac{8}{6}$	$\frac{6}{3}=\frac{8}{4}$	$\frac{8}{4}=\frac{6}{3}$	$3\bullet 8=6\bullet 4$

The names "flip", "switch", and "dial" are original with this writer, and are useful for helping students focus on distinctions. Introduction of these "transformations" is helped by performing these transformations with a specific arithmetic proportion such as $\frac{3}{4}=\frac{6}{8}$. Transformations performed on such a proportion yield results that are obviously true.

Again, **The Need: to help sensitize to several correct setups.**

Students must be cautioned that cross-multiplication only occurs across an equal sign, and the resulting products are set **equal** to each other.

Common Fractions: Operations Synthesis

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$+$	$898\frac{5}{6} = \frac{10}{12}$ $792\frac{11}{12} = \frac{11}{12}$	$-$	$438\frac{3}{4}$ $277\frac{11}{12}$

$$1690 + \frac{21}{12} \qquad 1690 + 1\frac{9}{12}$$

$$1690 + 1\frac{3}{4} \qquad 1691\frac{3}{4}$$

$$5\frac{5}{8} \times 2\frac{7}{9}$$

$$= \frac{45}{8} \times \frac{25}{9}$$

$$= \frac{\overset{5}{\cancel{45}}}{8} \times \frac{25}{\underset{1}{\cancel{9}}}$$

etc.

$$6\frac{1}{9} \div 1\frac{5}{6} =$$

Circle Attribute Schema

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Whenever a student is daunted by a problem involving a cone, cylinder, semicircle, hemisphere, etc., he or she should be instructed to complete these "circle blanks" and then re-read the problem.

AREA	RADIUS	DIAMETER	PERIMETER = CIRCUMFERENCE
A	R	D	P = C
πr^2	r	2r	πD

The terms are arranged in the indicated order because

- area relates most directly to radius
- radius relates directly to both area and diameter
- diameter relates directly to both radius and perimeter
- perimeter relates most directly to diameter

The above perceptual/relational schema applies principles of cognitive psychology demonstrated by Bower with earth science terms. Perceptual and/or relational schemas such as the above should be reviewed and applied briefly each day until students have mastery.

In the USA, the circle does not include the interior; the circle is only the points around the rim. In Europe, and in primary grades in the USA, the circle includes the interior. In the USA beyond primary grades, the term *disk* is used for the circle + interior.

Origin of π

For any size circle, $\pi = \dfrac{C}{D}$

This is demonstrated with round lids of varied color and size, and masking tape, with students working in pairs per teacher leading, drawing conclusions as pairs and then as a class. This discovery must be reinforced daily with review discussion, applications, and solving the tennis-ball problem.

Polygon Congruence
Distinction and Discrimination

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Congruent polygons or solids have the same *size* and shape.

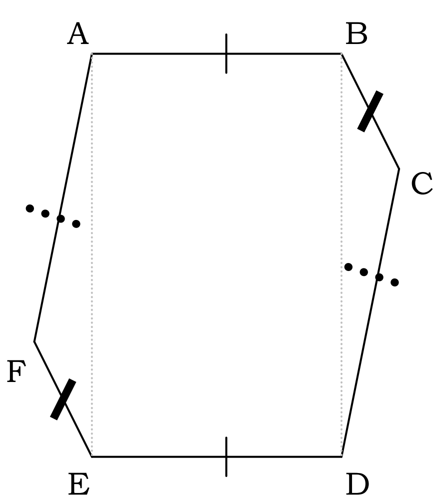
Similar polygons have the same shape.

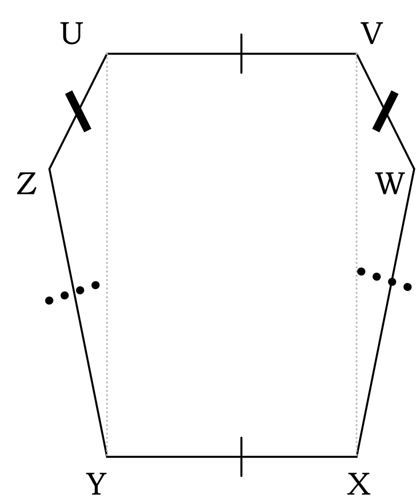
The simple definitions above are standard and sufficient for grade school students and great mathematicians alike. Confusion arises, however, beyond the point of definition, when ideas such as perimeter and area are introduced. *Further learning causes confusion*, say the learning scientists, and this connection helps students confront such confusion in a manner that strengthens understanding.

The first common confusion follows the logical error of accepting the converse of a statement without sufficient thought. A correct **principle** follows from the definition above: *Congruent polygons have the same area and perimeter.* Since this **principle is not a definition**, the converse must be examined carefully. The erroneous converse is this: "Polygons are congruent if areas and perimeters are equal." A counter-example that disproves this converse can be found in the illustration below.

The second common confusion is induced from the study of congruent triangles. Because of the congruence postulates, students are tempted to think that any two polygons, not just triangles, are congruent if all sides and angles are congruent.

An example is shown here wherein all sides (and therefore perimeters) are congruent, as are all angles (and area). The figures clearly are not congruent, as they lack the same shape. One is only line-symmetric, and the other is only point-symmetric.





Permutation and Combination Concept Development

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This writer first encountered permutations and combinations in an upper-division probability and statistics course in 1967, and then spent ten more years getting the ideas straight. Students in middle grades classes, by contrast, are now expected to make the distinction and count the possibilities. Confusion expressed by one of the teachers at a training workshop in summer 1999 motivates the formal discussion that follows, based on an organization used by in-service and pre-service audiences to help develop a conceptual understanding.

Definitions: A *permutation* is an ordered or arranged subset; a *combination* is an unordered subset. (The reader is expected to get nothing from those definitions at this point.)

Books:

Permutation. The boss is coming over for dinner and an impressive bookshelf is needed. Ten books of varying colors, sizes, and topics are available, with room on the shelf for six. *Different* arrangements (orders) of each subset of six will look different -- the heights might decrease from left to right, or increase; perhaps the tallest should be in the middle, and so on. How many six-book arrangements (*permutations*) are possible?

Combination. My wife has a list of ten book titles that she would like for Christmas. I will buy six of these books, and put the books in a box. Each gift-collection of six books (*combination*) will be the *same* for her no matter what order she takes them from the box. How many gifts are possible?

(The reader should still not expect to understand anything yet.)

Officers in a Club

Permutation. The mathematics club has ten members. The *different* offices are President, Vice-President, and Bouncer. If Alicia is President, Bill is Vice President, and Chuck is Bouncer, we have a different slate of officers from that if Bill is President, Alicia is VP, and Chuck is Bouncer. How many arrangements (*permutations*) are possible if all ten members are eligible?

Combination. The mathematics club has a cleanup committee consisting of Alice, Bill, and Chuck, all with the *same* committee membership. How many such cleanup committees (*combinations*) are possible if all ten members are eligible?

(The reader might begin to understand at this point, but if not, just keep on reading.)

Alphabet Letters

Permutation. The word *was* is different from the word *saw* because of the arrangement (order) of letters. How many different words (*permutations*) are possible if each letter of the alphabet may be used once at most?

Combination. A Wheel-of-Fortune contestant spins the wheel; the marker stops at a spot that says, "Pick three letters." The order that the contestant calls them out is not important; Vanna will still turn them all before the game proceeds. How many three-letter guesses (*combinations*) are possible from the eligible alphabet letters? Alternative: differing letters distributed in a Scrabble game.

(The reader should be catching on. The summary chart at the end might be helpful.)

Prizes

Permutation. The first-place winner gets $500; second-place gets a *different* $300; third-place gets a *different* $100. The winnings differ if the first-place and second-place winners trade places. How many ordered sets of place winners (*permutations)* are possible from 50 entries in the raffle?

Combinations. The three top winners in a raffle each get the *same* $400; the order doesn't matter. How many sets of top winners (*combinations)* are possible from 50 entries?

Digits

Permutation. The number 452 is *different* from the number 524 because of the different order. How many different three-digit numbers (*permutations*) are possible if the digits are chosen from 1 through 9?

Combination. In poker and other card games, the order that cards are dealt to each player doesn't matter; the player arranges and plays cards without any attention to the order in which the cards were received. If a player is dealt a four, then a five, and then a two, this is the *same* as being dealt a five, then a two, then a four. How many different poker hands (*combinations*) are possible if each hand consists of five cards dealt from a deck of 52 cards?

SUMMARY CHART

Model	**Permutation**	**Combination**
Books	on a shelf	in a gift box
Officers	distinct offices	committee
Letters	in a word	in a set
Prizes	distinct ranking	top tier
Digits	in a number	in a hand of cards

Quiz: Thirty-three ice cream flavors are available. Three are to be chosen for either a cone or a smoothie. (a) Is the cone a permutation or a combination? (b) How about the smoothie? (c) Does order matter to children in a lunch line? (d) Does order matter to children grouped about a piñata?

Mathematics Language and Details
(High school level?)

	Permutation	**Combination**
notation	$(3, -1) \neq (-1,3)$	$\{3, -1\} = \{-1,3\}$
abbreviation	${}_nP_r$	${}_nC_r$
idea	"n things <u>arranged</u> r at a time"	"n things <u>chosen</u> r at a time"
formula*	${}_nP_r = \dfrac{n!}{(n-r)!}$	${}_nC_r = \dfrac{n!}{(n-r)!r!}$

*The distinction between the two formulas is this: in the Combinations denominator, we divide out the number of rearrangements that are possible for each set, that number being $r!$.

Permutation specialties

Letters that repeat in a word (such as *Mississippi*): divide out repeaters.

Example: *Mississippi* has 11 letters, with i and s appearing four times each, and p appearing two times. The number of permutations using all 11 letters is

$$\frac{11!}{4!\,4!\,2!}$$

If n people sit in a circle, the number of permutations = $(n - 1)!$

If n keys are arranged in a ring, the number of permutations is $\dfrac{(n-1)!}{2}$.

Three Essential Basics for Radicals

1. $\sqrt{3} \bullet \sqrt{3} = 3$
2. $\sqrt{3} \bullet \sqrt{2} = \sqrt{6}$
3. $\sqrt{5} + \sqrt{5} + \sqrt{5} = 3\sqrt{5}$

Students can verify these, and check/disprove propositions such as

“Does $\sqrt{3} + \sqrt{2} = \sqrt{5}$?”

SET RELATIONSHIPS: D.I.C.E.S.

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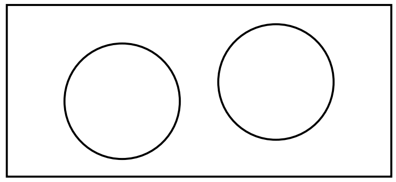
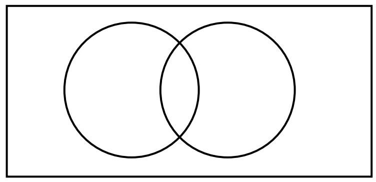
Disjoint

Probability:
"mutually
exclusive events"



$A \cap B = \varnothing$

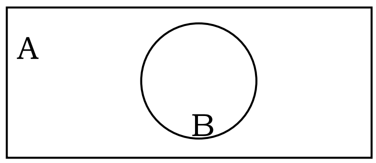
Intersecting



$A \cap B \neq \varnothing$

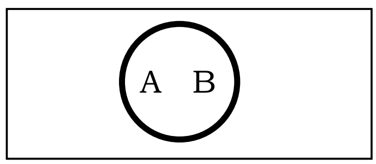
Complementary

Probability:
"complementary
events"

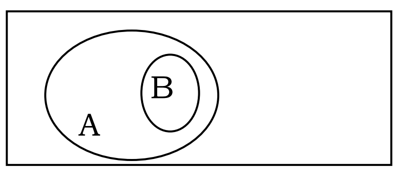


$A \cup B = U$
$A \cap B = \varnothing$

Equal



Subset



$B \subseteq A$

Core Concepts in Mathematics

Per music educators, music has a few core elements -- tone, pitch, timbre, and harmony, perhaps. By contrast, a global description of music might be "that which asks and answers questions with sound."

A global description of mathematics might begin with mathematics being the study of invariance, with number and form as basic elements. Core elements or concepts comprise a somewhat larger list, but the following list is stingy. Other ideas are embodiments of these concepts.

1. order
2. number
3. set
4. function
5. properties of operations and relations
6. classification/relationship
7. fraction/ratio
8. limit
9. operation

SET RELATIONSHIPS: D.I.C.E.S.

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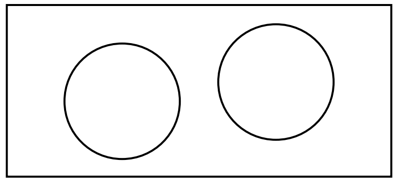
Disjoint

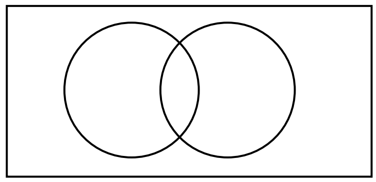
Probability: "mutually exclusive events"



$A \cap B = \varnothing$

Intersecting

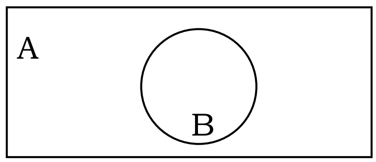
Probability: "independent events (perhaps)"



$A \cap B \neq \varnothing$

Complementary

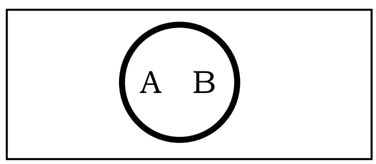
Probability: "complementary events"



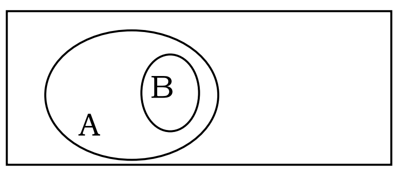
$A \cup B = U$

$A \cap B = \varnothing$

Equal



Subset



$B \subseteq A$

Core Concepts in Mathematics

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A music educator will state that there are but a few core elements of music. Among these are tone, pitch, timbre, and harmony. By contrast, a global description of music might be "that which asks and answers questions with sound, that sound consisting of tone, pitch, timbre,...."

A global description of mathematics might include the idea that mathematics is the study of invariance, with *number* and *form* as basic elements. Core elements or concepts comprise a somewhat larger list, but the following list is stingy. Other ideas are sub-embodiments of these key concepts.

order
number
set
function
classification/relationship
limit
fraction/ratio
operation
solution, root, solution set, zero
properties of relations and operations

Poster:

Core Concepts in Mathematics

order

number

set

function

classification/relationship

fraction/ratio

operation

limit

solution, root, solution set, zero

properties of relations and operations