Trigonometric Identities Connections

The Six Basics

$\sin(u+v) = \sin u \cdot \cos v + \sin v \cdot \cos u$

$\sin(u-v) = \sin u \cdot \cos v - \sin v \cdot \cos u$

$\cos(u+v) = \cos u \cdot \cos v - \sin u \cdot \sin v$

$\cos(u-v) = \cos u \cdot \cos v + \sin u \cdot \sin v$

The tanθ formulas below come from sinθ/cosθ used on the preceding formulas.

$$\tan(u+v) = \frac{\tan u + \tan v}{1 - \tan u \tan v}$$

$$\tan(u-v) = \frac{\tan u - \tan v}{1 + \tan u \tan v}$$

Double-Angle

Use sum formulas; let $u = v$

$\sin 2u = 2 \sin u \cos u$

$\cos 2u = \cos^2 u - \sin^2 u$

$= 2\cos^2 u - 1$ *

$= 1 - 2\sin^2 u$ *

$$\tan(2u) = \frac{2\tan u}{1 - \tan^2 u}$$

***These come from substitution with $\sin^2\theta + \cos^2\theta = 1$**

Power-Reducing

From the * formulas in the previous column:

$$\sin^2 u = \frac{1 - \cos 2u}{2}$$

$$\cos^2 u = \frac{1 + \cos 2u}{2}$$

$$\tan^2 u = \frac{1 - \cos 2u}{1 + \cos 2u} \quad *$$

*** From where?**

Half-angle

Replace u with $\frac{u}{2}$ in previous column, and do square root.

$$\sin\frac{u}{2} = \pm\sqrt{\frac{1 - \cos u}{2}}$$

$$\cos\frac{u}{2} = \pm\sqrt{\frac{1 + \cos u}{2}}$$

$$\tan\frac{u}{2} = \pm\sqrt{\frac{1 - \cos u}{1 + \cos u}}$$

Product-to-Sum

Multiply third formula above by –1; add result to fourth formula and solve for $\sin u \sin v$.

$$\sin u \cdot \sin v = \frac{1}{2}[\cos(u - v) - \cos(u + v)]$$

Add third and fourth formulas above and solve for

$$\cos u \cdot \cos v = \frac{1}{2}[\cos(u - v) + \cos(u + v)]$$

Add first and second formulas above and solve for

$$\sin u \cdot \cos v = \frac{1}{2}[\sin(u + v) + \sin(u - v)]$$

Multiply second formula above by –1; add result to first formula and solve for $\cos u \sin v$.

$$\cos u \cdot \sin v = \frac{1}{2}[\sin(u + v) - \sin(u - v)]$$

Sum-to-Product

Preliminary shoveling:

Let $x = u + v$ and $y = u - v$.

Add those two equations:

$x + y = 2u$

so $u = \frac{x+y}{2}$

Subtract those two equations:

$x - y = 2v$

so $v = \frac{x-y}{2}$

Now substitute these into the Product-to-Sum formulas and solve for the sum or difference:

$$\sin x + \sin y = 2\sin\left(\frac{x+y}{2}\right)\cos\left(\frac{x-y}{2}\right)$$

$$\sin x - \sin y = 2\cos\left(\frac{x+y}{2}\right)\sin\left(\frac{x-y}{2}\right)$$

$$\cos x + \cos y = 2\cos\left(\frac{x+y}{2}\right)\cos\left(\frac{x-y}{2}\right)$$

$$\cos x - \cos y = -2\sin\left(\frac{x+y}{2}\right)\sin\left(\frac{x-y}{2}\right)$$