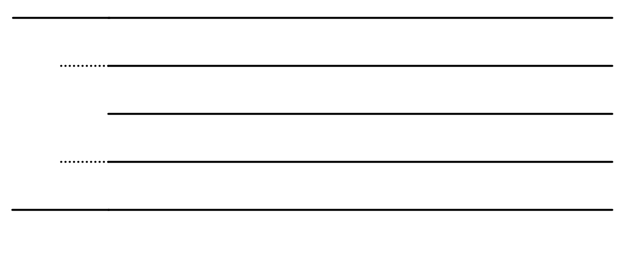
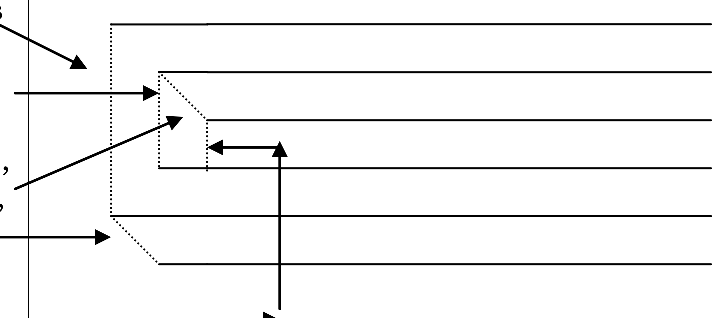
The Impossible E

Dr. Stan Hartzler Archer City High School

1. Draw six parallel lines, evenly spaced, as shown. The distance between the first and sixth should be less than the length of each.	
2. Estimate the distance between each adjacent pair of lines. Extend the first (topmost) and fifth lines by twice that estimated amount. New steps are dotted.	
3. Again estimate the distance between each adjacent pair of lines. Extend the second, fourth, and sixth lines by that estimated amount.	
4. Connect the left ends of segments 1 and 5. Connect the left ends of segments 2 and 4. With slanted segment, connect ends of 2 and 3, and of 5 and 6. Connect the end of 3 to 4 with segment perpendicular to 4.	
5. Add three ovals to right end as shown. Draw mice or other critters to enhance the alternating representation of space.	

The illusion is apparent if one looks at the left side and sees a visual representation of two “prongs” with a space between them. If the eyes follow the space to the right, that space becomes a solid third prong, between two others. The outer two still look three-dimensional, but for a different reason.

Purposes:

1. Drawing skill
2. Visual estimation
3. Spatial sense
4. Fun
5. Art appreciation (Escher) and problem-solving therein.

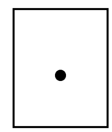
Steps for Drawing the Penrose Triangle

Dr. Stanley J. Hartzler   Archer City High School

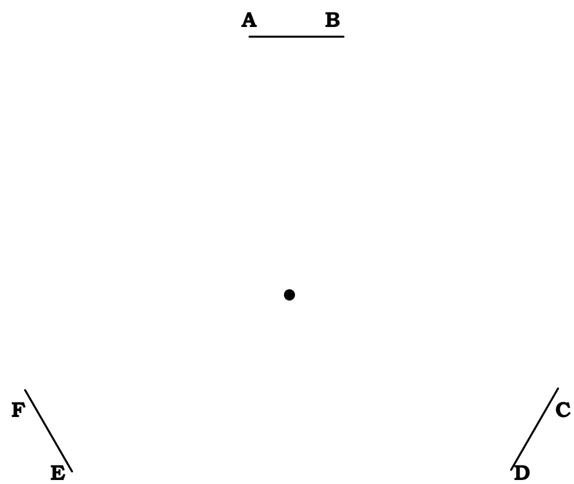
S*patial sense* or *visuospatial sense* is needed for learning mathematics and science. This sense is one of only two of Gardner's "intelligences" that has been confirmed by learning (cognition) scientists. Activities (origami and drawing of ambiguous figures) that help develop spatial sense should be used in classrooms. The purpose of this paper is to provide directions for one such ambiguous drawing.

Visuospatial sense, per Elizabeth Fennema, refers to ability to look at a two-dimensional drawing of a solid object and see it accurately in three dimensions. The mind automatically attempts such sensibility.

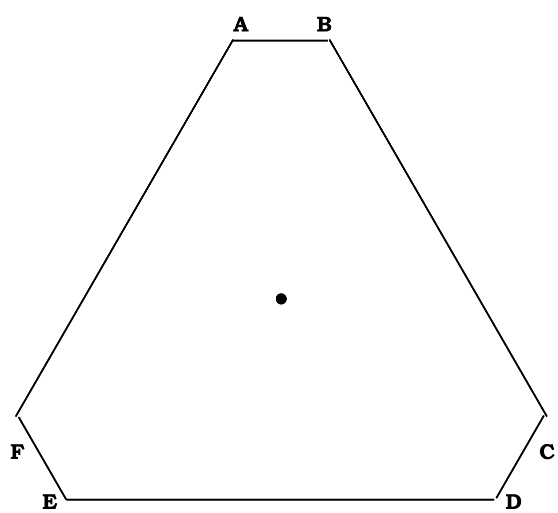
Step 1: Place a dot in the middle of a page of paper, and imagine this to be the center of a clock with hands.



Step 2: Draw $\overline{AB}$, $\overline{CD}$, and $\overline{EF}$ as shown, at positions of 12 o'clock, four o'clock, and eight o'clock, respectively.



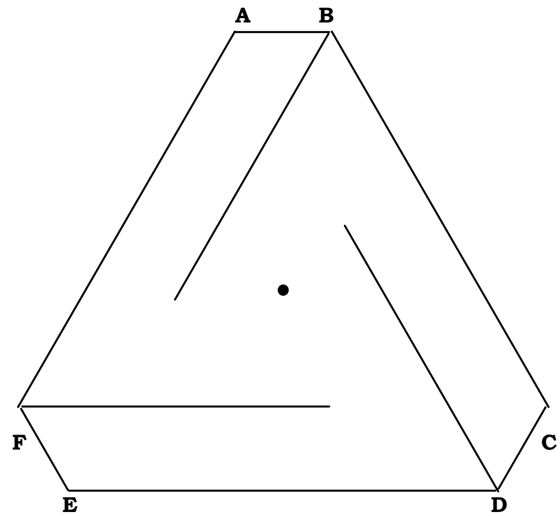
Step 3: Draw $\overline{BC}$, $\overline{DE}$, and $\overline{FA}$ as shown.



Step 4: Place pencil at point F, and begin to draw segment FC. Stop a little past halfway.

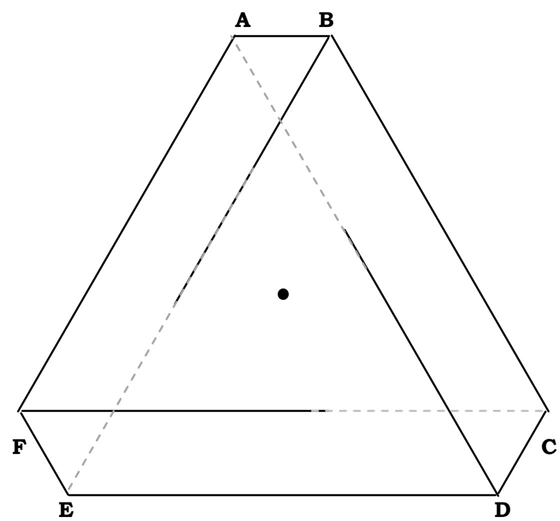
Next, place pencil at point D, and begin to draw segment DA. Stop a little past halfway.

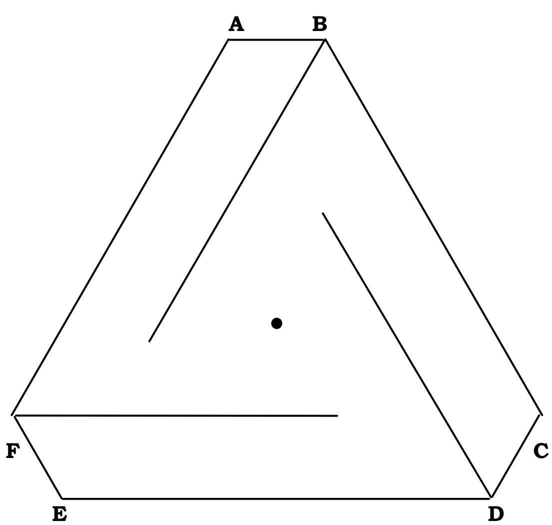
Last, place pencil at point B, and begin to draw segment BE. Stop a little past halfway.



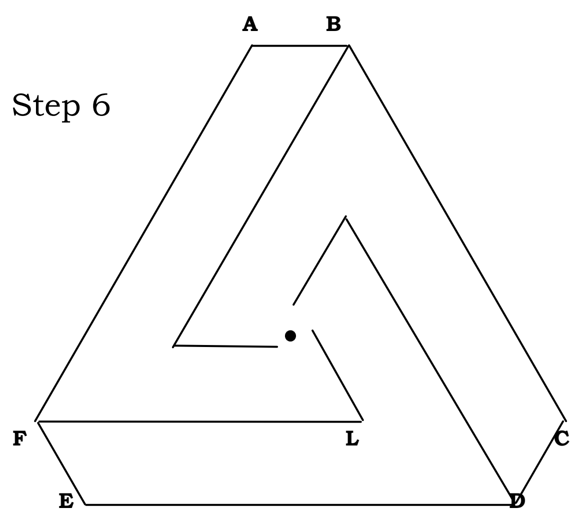
Step 5: Visualize the completion of segment FC. Visualize how much of FC would be between segments CB and DA. That length of FC must be mentally copied on the left of DA. Draw FL so that LC is bisected by DA. Do not draw the dotted line segment LC.

Repeat the same idea for segments DA and BE. Do not draw dotted segments.



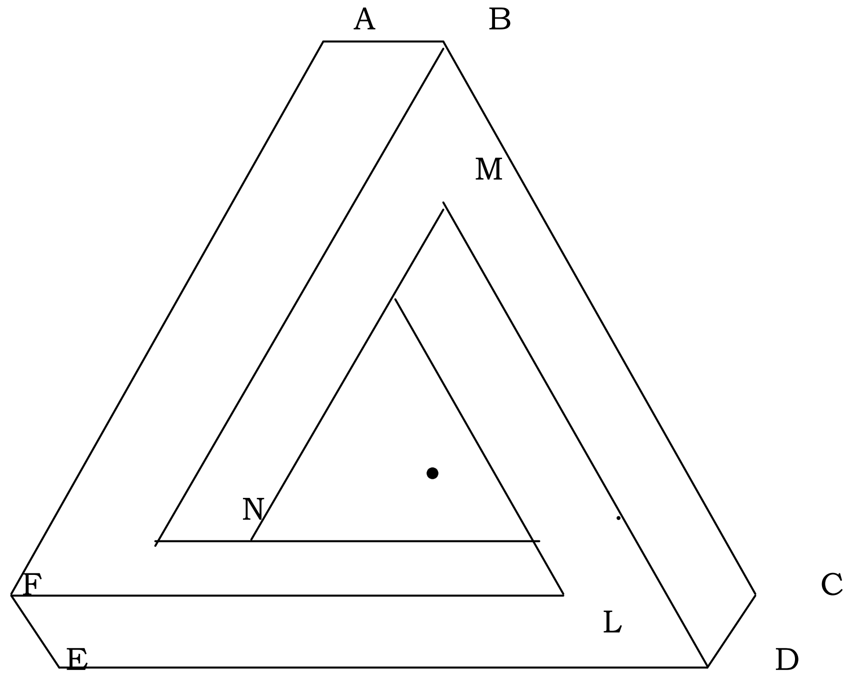


Step 6



Step 6. Place pencil at L and draw a segment parallel to segment CB and partial DA (as much of DA as is drawn) BUT only of a bit past the middle. Repeat for M and N.

Step 7: Continue drawing each of the three new segments until the pencil hits another of the three new segments. The Penrose Triangle is complete.



This triangle suggests a visual paradox. Looking only at the top two "bars" of the triangle, it appears that points A, B, and M (and another hidden point) are included in the intersection of two bars, with *the left-most bar being* ***in front of*** *the right-most bar*.

Next, if only the left and bottom bars are examined similarly, it appears that the left bar is behind the bottom bar. By transitivity, this would place *the bottom bar* ***in front of*** *the right bar*.

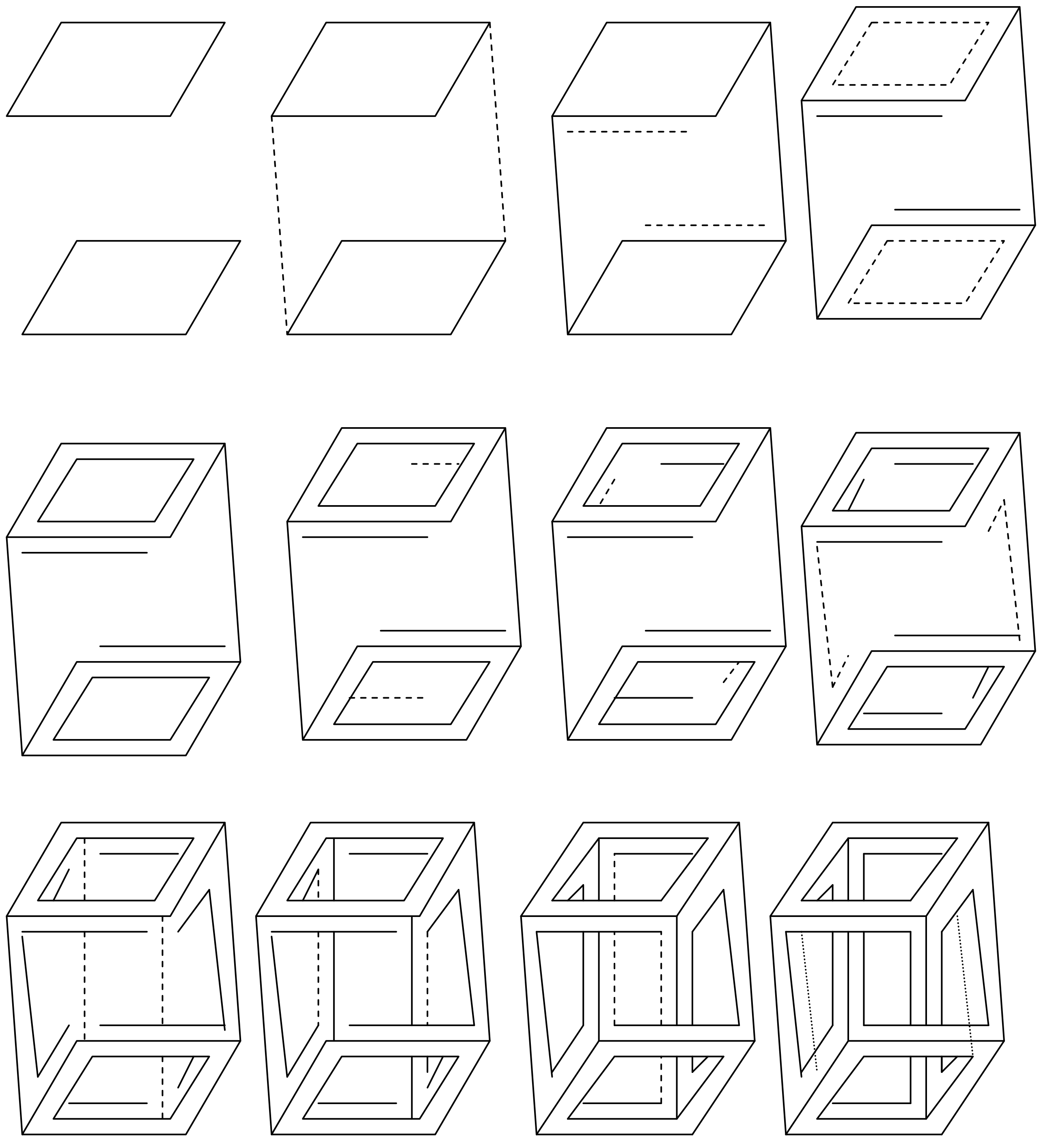
Finally, if only the bottom bar and the right bar are examined, it appears that *the bottom bar is* ***behind*** *the right bar*.

Thus, each bar is both in front of *and* behind each of the other two. As eye and mind attempt to make three-dimensional sense of this two-dimensional drawing, frustration will occur without any of this analysis.

Escher artfully involved four such triangles in his graphic, *Waterfall*.
Other such impossible figures used by artists include the Impossible E and the Impossible Cage. Such use involves problem-solving…

…as does *Belvedere* by Escher. Steps for drawing the Impossible Cage are on the next page. The main structure in *Belvedere* has supporting columns drawn deceptively, explaining a ladder paradox. On a bench outside the Belvedere, to the viewer's left, a person holds a basic version of the Impossible Cage.

Drawing the Impossible Cage



Soccer Shootout Problems
A Primary-Grade Introduction to Probability

When a soccer game ends in a tie, there is an overtime period. If the score is still tied after the overtime period, then a "shootout" occurs. Five players are selected from each team. These players each take turns shooting at the goal from the top of the goal arch while the other team's goalie defends the goal.

If the score is still tied after the first five from each team shoot, the next five from each team shoot, continuing on if needed with the third five, and, if needed, rotating through the top five again, and so on until the tie is broken.

Clearly, it is advantageous if a coach knows who the team's good shootout kickers are. Two such imaginary coaches have tested their teams with 24 practice kicks and recorded the results.

Their soccer teams have fifteen players each. The uniform numbers are listed below, along with the estimated probability of each player scoring a goal in a shootout situation.

	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19
T B	$\frac{19}{24}$	$\frac{3}{8}$		$\frac{11}{24}$	$\frac{3}{4}$	$\frac{1}{8}$	$\frac{13}{24}$	$\frac{5}{8}$	$\frac{1}{6}$	$\frac{11}{24}$		$\frac{1}{4}$	$\frac{11}{12}$	$\frac{7}{24}$	$\frac{1}{2}$		$\frac{5}{12}$		$\frac{2}{3}$
N J	$\frac{13}{24}$		$\frac{5}{6}$	$\frac{1}{12}$	$\frac{3}{8}$	$\frac{1}{4}$	$\frac{7}{12}$	$\frac{5}{24}$	$\frac{1}{2}$	$\frac{3}{8}$	$\frac{17}{24}$		$\frac{7}{8}$	$\frac{1}{3}$		$\frac{11}{24}$		$\frac{1}{2}$	$\frac{3}{4}$

Project 1 (any grade): Set up a shootout. Rank order the members of the Thunderbolts in terms of their shootout ability. When finished, rank-order the Ninjas.

Project 2 (any grade). Create models of players and teams. Get 30 wide-mouth canning jars, paper sacks, or other such containers, and 200 beads or marbles of one color and 200 more of another color. Label each jar wth the team name and the uniform number of the player it represents. Then put balls in the jars to model that player's ability to score during a shootout, per the following example, where blue balls represent goals and red balls represent misses:

Player #16 for the Ninjas will score 11 goals in 24 attempts. Put 11 blue marbles and 13 red marbles in the jar labeled Ninjas #16.

Do the same for each player on each team.

Project 3 (any grade). Use the model to show probability in action. Play a shootout. Two members of the class are chosen to draw a ball at random from the best five player-jars for each team. If the score is still tied after the first five, go on into another round with the second five's until the tie is broken. Record the final score.

Project 4 (any grade). Generate a data set. Assign class members to before-school shootouts, with two class members going through one shootout on a given day. Keep a record of wins and losses and scores. On the outside of each jar, record each player's actual goals and misses.

Related question: when each day's shootout is over, should the balls drawn out be put back or left out until all balls are drawn out for each player?

Project 5 (Grade 4 or above): Evaluate the model. At the end of the year, compare the theoretical probability assigned to each player with the actual number of goals scored by drawing out beads.

Project 6 (any grade): Do a superficial check for team balance. Give each player's probability a denominator of 24. Does the sum of the first five's favorable outcomes match that of the other team's first five? The second fives'? The third?

Advanced (secondary-grade) problem: Is this sufficient to show balance? Compare the number of first-five wins for each team.

Project 7 (any grade): Maintain perspective. Update (improve) selected probabilities every nine weeks to reflect improved skills. For example, the player with only three goals in 24 attempts has had nine weeks to practice, enough to improve. Pick three members of each team who had been among the weakest, and improve their tryout results by the same amount. Rank-order the teams again, and keep this nine-weeks data separate from that of the previous.

Project 8 (high school): Address a real-world complication. It was once the case that another tournament game was to be played after the Thunderbolt-Ninja Shootout. The officials called the game before everyone shot in the first shootout round because the Thunderbolts were ahead 3-1 with one player remaining to shoot for each team. Find the probability that under such time constraints,
the game will be declared over after just three players shoot, then after four players shoot, then after two players shoot.

Project 9 (high school): Find the probability that there will still be a tie after the first round of a shootout.

Answer to rank-order question: (Second and fourth rows are uniform numbers.)

rank	*1*	*2*	*3*	*4*	*5*	*6*	*7*	*8*	*9*	*10*	*11*	*12*	*13*	*14*	*15*
TB	**13**	**1**	**5**	**19**	**8**	**7**	**15**	**4**	**10**	**17**	**2**	**14**	**12**	**9**	**6**
P	$\frac{11}{12}$	$\frac{19}{24}$	$\frac{3}{4}$	$\frac{2}{3}$	$\frac{5}{8}$	$\frac{13}{24}$	$\frac{1}{2}$	$\frac{11}{24}$	$\frac{11}{24}$	$\frac{5}{12}$	$\frac{3}{8}$	$\frac{7}{24}$	$\frac{1}{4}$	$\frac{1}{6}$	$\frac{1}{8}$
NJ	**13**	**3**	**19**	**11**	**7**	**1**	**9**	**18**	**16**	**5**	**10**	**14**	**6**	**8**	**4**
P	$\frac{7}{8}$	$\frac{5}{6}$	$\frac{3}{4}$	$\frac{17}{24}$	$\frac{7}{12}$	$\frac{13}{24}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{11}{24}$	$\frac{3}{8}$	$\frac{3}{8}$	$\frac{1}{3}$	$\frac{1}{4}$	$\frac{5}{24}$	$\frac{1}{12}$

What Science Students Need to Know About Fractions

Dr. Stan Hartzler Archer City High School

1. Numerator tells how many; denominator tells what kind*
   *connect to chemistry

2. Four meanings of a fraction
   a) fractional part of a whole
   b) fractional part of a group
   c) ratio
   d) division

3. Forms: $3 \div 7$ means $\frac{3}{7}$ means $7\overline{)3}$, etc.

4. Unit fraction: numerator = 1

5. Speed is fraction with time on the bottom

6. Dimensional analysis: conversion fractions = 1

7. Weight per unit volume: do the division so that denominator = 1

8. Per cent is fraction with denominator of 100

9. Variation: direct, inverse, joint; rate of work;
   hen-and-a-half lays an egg-and-a-half in a day-and-a-half

10. Proportion is the equality of two ratios.
    Flip
    Switch
    Dial
    Cross-multiply

11. Calculator: how to enter $\frac{13.6 \times 88}{.053 \times 7020}$ into a calculator correctly.

12. If the numerator of a fraction increases while the denominator remains the same, then the fraction's value increases. If the denominator's value increases while the numerator remains the same, the value of the fraction decreases. If both numerator and denominator increase, we can't tell how the new value compares automatically. The same is true if both decrease.

This is useful when considering the effects on intensity or density, or other such ratios, of adjustments made to either variable.

SIMPSON'S PARADOX

Dr. Stan Hartzler Archer City High School

The examples here are courtesy of Dr. Peter John of the Mathematics Department at The University of Texas at Austin, 1980.

Mathematics survey textbooks, and chapters and sections thereof, are often titled "The Use and Misuse of Statistics." This paper is directed at the misuse category. It is usually omitted in such chapters.

Example 1: There are two treatments given to a disease (dandruff?), with both treatments given to men and women. The percent cured is figured by *number cured* divided by *number treated* and the quotient is written as a per cent.

	TREATMENT A	**TREATMENT B**
MEN	$\frac{20}{100} = 20\%$	$\frac{50}{210} = 24\%$
WOMEN	$\frac{40}{60} = 67\%$	$\frac{15}{20} = 75\%$

At this point, it appears that Treatment B is better for both. But: Treatment A, all totaled, cured 60 out of 160 people, or 38%, while Treatment B cured only 65 out of 230 people, for 28%. Now Treatment A appears to be the best.

Now for an example which (I hope) demonstrates how statistics can be misused to get people to think and act and vote and contribute strongly and wrongly. This example is deliberately close to home and personal bias.

It may be well and good for me to inform you here that I, Hartzler, was delivered at birth by a woman doctor, and that my mother and three sisters were career-minded before husband-minded, thus making me astonished and sad when I began dating girls whose primary source of self-image was what the men thought of them. That aspect at least of equality-of-sexes has my support.

Example 2: A company employs 50 men and 50 women, with the male salary average $15,920 and the female salary average $14,000.

Discrimination? No.

YEARS OF SERVICE	**MEN**		**WOMEN**	
	number	**salary**	**number**	**salary**
LESS THAN 5	10	$10,000	40	$12,500
MORE THAN 5	40	$17,400	10	$20,000

The issue is that women are newer to the job market, and have less overall experience, which is one thing companies pay for.

It might appear by the above breakdown that the men are being discriminated against. Not so: perhaps the women are better trained, which is ANOTHER thing companies pay for.

The attention of future teachers is needed. Simpson's Paradox can, and should, be taught to grade-school students as soon as they can work with per cents.

But most of all: when people throw statistics at us to spur us on to action and agitation, we should be careful.

Which way is best? In the first example, Cure B is best. It is always desirable to look at statistical breakdowns by category or categories. When a pattern exists (or does not exist) across different categories, one may be generally more confident when making inferences about truth and then decisions.

Why do students need to learn mathematics? What can we communicate to them that may motivate them?

Students help themselves by being good students in mathematics, by preparing themselves for a wide variety of good-paying jobs with short working hours, good security, and better vacation options. Students help their communities and nation by being productive workers and managers. Mathematics helps here also.

Students help themselves and communities and nation by being intelligent citizens. Those aware of Simpson's Paradox may be able to address community issues where statistics appear to show a problem or solution, but where a somewhat more detailed breakdown of data would show an opposite problem or a different solution.

The mathematically-literate citizen will use mathematics for the common good, and will be able to help when others might attempt to misuse mathematics for private fraud. For these reasons, mathematics teachers and students should be motivated to succeed.

"AMAZING GRACE" AND CANTOR

Dr. Stan Hartzler Archer City ISD

A definition of *infinite set* that is helpful in the study of Cantor's Theorem is this:

A set is infinite if it is equivalent to a proper subset of itself.
(Berlinghoff & Grant, 1992.)

An example of such a set, and application of the definition, is suggested in the last verse of the hymn, "Amazing Grace":

When we've been there ten-thousand years
Bright shining as the sun
We've no less days to sing God's praise
Than when we've first begun.

The verse suggests the following demonstration of equivalence:

$$
\begin{array}{ccccccccccc}
 & \{ & 10{,}001 & , & 10{,}002 & , & 10{,}003, & \ldots, & n + 10{,}000 & , \ldots & \} \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
N = & \{ & 1 & , & 2 & , & 3 & , \ldots, & n & , \ldots & \}
\end{array}
$$

Cantor's work began to interest other mathematicians in the late 1800's. "Amazing Grace" was written by John Newton in the late 1700's. The last verse written above is credited by *The Baptist Hymnal* (1991) to an anonymous writer around 1790; *Crusader Hymns and Hymn Stories* (1967) has this last verse ascribed to John P. Rees (1828-1900).

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Creativity/Problem-Solving per Cognitive Principles for Students and Teachers

Four steps per Poincare, Wallas, and others:

1. Saturation
   a. Mastery of background skills to level of successful teaching.
   b. Modeling of problem-solving, including playful attitude.
   c. Polya's list of strategies and corresponding examples.
   d. Selection and careful reading/re-reading of problem.
   e. Initial attempts to solve problem
2. Incubation
   Time away from the problem; sleep

[2.5 Possible revisiting to further stimulate incubation. Review of old attempts, new attempts, more time away, and repeating patiently and with a playful attitude.]

3. Illumination
   The solution occurs to the solver; "Aha!"; the light goes on.
4. Verification

Two more steps per common pedagogy in 1990's:

5. Extension
   Solver modifies problem to add complexity, new elements, change quantification, etc.
6. Application
   A practical application is proposed.

Teacher Creativity: A Related Phenomenon

Why veteran teachers are prone to new content and pedagogical discoveries, linkages, problem-creations, and other moments of superb creativity:

1. Saturation occurs when the same course is taught each semester for several years.

2. Incubation occurs during Christmas break, summer break, etc.

3. Illumination just happens as a natural effort of the brain to organize the increasing associations in, and distinctions between, features of what is taught.

4. Verification includes sharing with students and colleagues.

A Connecting Strategy for Ratio, Proportion, Per Cent Problems

Dr. Stan Hartzler Archer City High School

A table helps the mind organize data and understand problem situations. Cognitive psychology supports the proposition that the mind attempts to organize perceptual input, that memory is aided by perceptual and conceptual schemas, and that achievement is improved very significantly and efficiently when schemas are used for initial learning and repeated in practice and recall.

A generic table such as the following may assist students in a variety of ratio-proportion problems, with direct extensions to per cent, and to atomic weight problems in chemistry.

item	ratio	actual

A basic ratio problem would be step as follows:

1. For every six WWE fans, there are seven NASCAR fans. If there are 84 WWF fans, how many NASCAR fans are there?

item	ratio	actual
WWE	6	84
NASCR	7	X

The table suggests a proportion perceptually: $\frac{6}{7} = \frac{84}{X}$.

Note A: Solution by cross-multiplication should include the caution that cross-multiplication is only valid over an equals symbol, and that the products must be written equal to each other for solution.

A more advanced ratio problem involves the concept of the total individuals involved.

2. For every eleven corn kernels there are six wheat kernels in the mess. If there are 206 kernels in all, how many are corn?

item	ratio	actual
corn	11	X
wheat	6	
total	17	206

Note B: The data for wheat is incomplete but is no longer interesting once the simplest proportion is evident: $\frac{11}{17} = \frac{X}{206}$.

Note C: The insertion of a table entry for TOTAL is an example of an initiative or innovation that students need to be encouraged to make.

Note D: Other methods of solution are common, of course; what is suggested here is a solution that is uncomplicated once the organization is complete.

3. Three out of every 16 were oversized.

item	ratio	actual

If there were 48 in all, how many were not oversized?

oversize	3	
not	13	*X*
TOTAL	16	48

Note E: Another variety of initiative was needed.
Note F: Example 3 might have been worded, "Three-sixteenths were oversized…" and this strategy may be used for such problems, and perhaps can be introduced to students in earlier grades than those normally used to introduce the examples given here.

4. Forty-one per cent of the pans were dented. If 323 pans were not dented, how many pans were there in all?

item	ratio	actual
dented	41	
not	59	323
TOTAL	100	*X*

Note G: Two initiatives are shown: the ratio total, and the complementary 59%

Per cent of increase in the following setting can be tricky, but the suggested schema provides clarity and efficient solution.

5. The successful solutions increased by 110%. If there were 73 successes at the beginning, how many were there at the finish?

item	ratio	actual
start	100%	73
increase	110%	
finish	210%	*X*

Note H: Initiative is needed in identifying the original amount as 100%.
Note I: The concept of per cent of increase is clarified.

The problem may also be modified by providing the actual increase, or in asking for the actual increase.

6. The number of registered voters increased from 671 to 712. What is the per cent of increase?

item	ratio	actual
start	100%	671
increase	*X*	41
TOTAL		712

Note J: The insertion of 100% as the original amount is based on the definition of per cent of increase as well as the per cent of decrease. Each is defined in business and science as the amount of change divided by the original amount.

7. The number of holdouts decreased from 30 to 23. What is the per cent of decrease?	item	ratio	actual
	start	100%	30
	decrease	X	7
	TOTAL		23

Applications to some basic chemistry problems require reference to atomic numbers from the Periodic Table of Elements.

8. In 78 grams of H_2SO_4, how many grams are there of sulfur?	*element*	*atomic wt**	*atoms @*	*ratio*	*actual gm*
	Hydrogen	1	2	2	
	Sulfur	32	1	32	X
	Oxygen	16	4	64	
TOTAL →	H_2SO_4			98	78

Thus, $\frac{32}{98} = \frac{X}{78}$, etc.

* Number of basic particles (electrons or protons or neutrons) in each atom.

Ten years ago, 70% of the mathematics portion of the ACT college board exam consisted of problems that could be solved using ratio and proportion. The importance of this topic was thus established for getting into higher education, and for succeeding in the mathematics portions of business, science, and so on.

Connection: Sine Values and Electromagnetic Wave Forms

Dr. Stan Hartzler Archer City High School

1. Draw a quarter-circle on a large sheer of paper, with radius one meter.

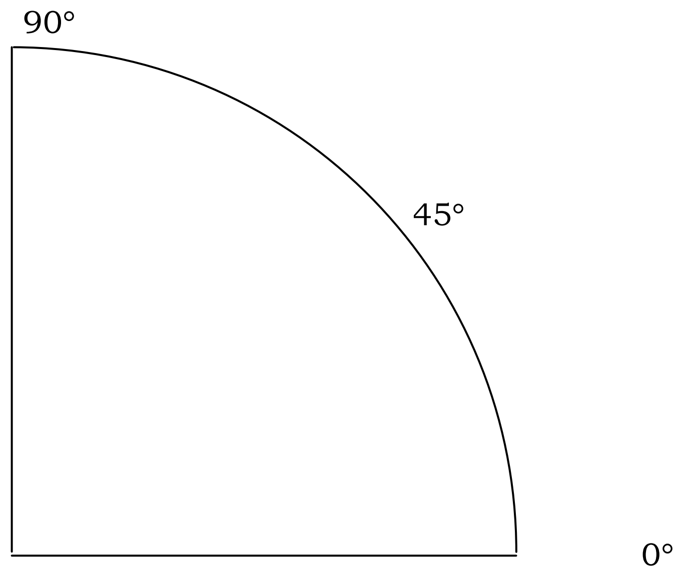
2. Mark angles 0°-90°.

3. Make a chart of angle measures 0°-90°.

4. Measure vertical distance from each angle mark to horizontal segment. Record in chart as meters.

5. Graph as shown below table.

6. Note the shape. Show copies and extensions.



Angle	1°	2°	…	30°	31°	…	53°	54°	55°	…
Distance	.017	.349	…	.500	.515	…	.799	.809	.819	…

This shape is the shape of light waves, radio waves, x-rays, TV waves, etc.

Why? That question has attracted the attention of philosophers.

Let students experience that question.